Representation type of Jordan algebras

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Jordan algebras arose from the search of "exceptional" setting for quantum mechanics.

The "Copenhagen model":

- the physical observables are Hermitian matrices (or operators on Hilbert space);
- the basic operations are multiplication by $\lambda \in \mathbb{C}$, addition, multiplication of matrices, adjoint operator.

Most of matrix operations are not observable!

In 1932 the German physicist Pascual Jordan proposed a program to discover *"a new algebraic setting for quantum mechanics"*,

Jordan has chosen a new observable operation called **quasi-multiplication**

$$x \cdot y := \frac{1}{2}(xy + yx).$$

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Definition

A **Jordan algebra** J consists of a real vector space equipped with a bilinear product $x \cdot y$ satisfying the commutative law and the Jordan identity

$$x \cdot y = y \cdot x,$$
$$(x^2 \cdot y) \cdot x = x^2 \cdot (y \cdot x), \ \forall x, y \in J. \quad (1)$$

A Jordan algebra is **formally real** if

$$x_1^2 + \cdots + x_n^2 = 0 \Longrightarrow x_1 = \cdots = x_n = 0.$$

Any associative algebra A over $\mathbb{R}$ gives rise to a Jordan algebra A^+ under quasi-multiplication: the product $x \cdot y$ is clearly commutative and satisfies the Jordan identity.

A Jordan algebra is called **special** if it can be realized as a Jordan subalgebra of some A^+. For example, if A has an involution $\star$ then the subspace of hermitian elements $H(A, \star) = \{x^\star = x \mid x \in A\}$ forms a special Jordan algebra.

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These hermitian algebras are the archetypes of all Jordan algebras. It is easy to check that the hermitian matrices over $\mathbb{R}$, $\mathbb{C}$ and the quaternions form special Jordan algebras that are formally real.

One obtains another special formally real Jordan algebra (called **spin factor** $JSpin_n$ or **Jordan algebra of Clifford type**) on the space $\mathbb{R}1 \oplus \mathbb{R}^n$, when $n \geq 2$, by making 1 acts as a unit and defining $v \cdot w = \langle v, w \rangle 1$, where $v, w \in \mathbb{R}^n$ and $\langle \, , \, \rangle$ is an inner product in $\mathbb{R}^n$.

What he physicists were looking for were Jordan algebras where there is no invisible structure xy governing the visible structure $x \cdot y$. A Jordan algebra $\mathcal{J}$ is called **exceptional** if it is not special, i.e. does not result from quasi-multiplication.

Jordan hoped that by studying finite-dimendional Jordan algebras he could find families of simple exceptional algebras E_n, parametrized by natural numbers n, letting n to infinity would provide a suitable infinite-dimensional home for quantum mechanics.

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In 1934 Jordan, John von Neumann and Eugene Wigner showed that every finite-dimensional formally real Jordan algebra can be written as a direct sum of so-called simple ones, which are four infinite families, together with one exceptional case:

- The Jordan algebra of $n \times n$ self-adjoint real matrices, complex matrices and quaternionic matrices;
- The Jordan algebra of Clifford type corresponding to usual inner product in $\mathbb{R}^n$;
- The Jordan algebra $H_3(\mathbb{O})$ of 3×3 self-adjoint octonionic matrices, (an exceptional Jordan algebra called the **Albert algebra**, since A.Albert showed that it is indeed exceptional).

There was only one exceptional algebra in the list!!! $H_n(\mathbb{O})$, $n > 3$ is not Jordan, while $H_2(\mathbb{O}) \simeq JSpin_9$. The lone $H_3(\mathbb{O})$ was too isolated to give a clue as to possible existence of inf-dimensional exceptional algebras.

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Let $\mathbf{k} = \bar{\mathbf{k}}$, char$\mathbf{k} = 0$. A **Jordan algebra** is a commutative $\mathbf{k}$-algebra $(J, \cdot)$ satisfying Jordan identity

$$(x^2 \cdot y) \cdot x = x^2 \cdot (y \cdot x) \quad x, y \in J. \qquad (2)$$

Example

1. Denote $H_n(A) = H(M_n(A), \bar{j}) \subset M_n(A)^+$.
2. Let f be a symmetric bilinear form on E, then $J(E, f) = E \oplus \mathbf{k}$ with respect to

$$(\alpha + x) \cdot (\beta + y) = \alpha\beta + f(x, y)1 + (\alpha y + \beta x),$$

   $\alpha, \beta \in \mathbf{k}$, $x, y \in E$ is a Jordan algebra of Clifford type.
3. Let $\mathbb{O}$ be an Caley algebra over $\mathbf{k}$, then $\mathcal{A} = H_3(\mathbb{O})$ is called the **Albert algebra**

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Suppose M is a **k**-vector space equipped with two bilinear mappings

$$l, r : J \otimes M \to M \quad l : (a, m) \to am, \quad r : (a, m) \to ma,$$

and define a product on $\Omega = J \oplus M$

$$(a_1 + m_1) \circ (a_2 + m_2) = a_1 \cdot a_2 + a_1 m_2 + m_1 a_2.$$

M is a **Jordan bimodule** for $J \Leftrightarrow \Omega = (\Omega, \circ)$ is a Jordan algebra.
Equivalently a linear map $\rho : J \to \mathrm{End}_k M$, $\rho(a)m = am$ defines a **(bi)representation** if for all $a, b \in J$

$$[\rho(a), \rho(a \cdot a)] = 0,$$

$$2\rho(a)\rho(b)\rho(a) + \rho((a \cdot a) \cdot b) = 2\rho(a)\rho(a \cdot b) + \rho(b)\rho(a \cdot a).$$

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For any J the category of **k**-finite dimensional J-bimodules is equivalent to the category of finite dimensional left U-modules for some associative algebra called the **universal multiplicative enveloping algebra** $U = U(J)$.

The algebra U can be constructed as a quotient of the tensor algebra $F(J)/R$, where R is an ideal generated by the elements:

$$\begin{aligned} & a_1 \otimes a_2 \otimes a_3 + a_3 \otimes a_2 \otimes a_1 + (a_1 \cdot a_3) \cdot a_2 \\ & \quad - a_1 \otimes a_2 \cdot a_3 - a_2 \otimes a_1 \cdot a_3 - a_3 \otimes a_1 \cdot a_2, \\ & a_1 \otimes a_2 \cdot a_3 + a_2 \otimes a_1 \cdot a_3 + a_3 \otimes a_1 \cdot a_2 \\ & - a_2 \cdot a_3 \otimes a_1 - a_1 \cdot a_3 \otimes a_2 - a_1 \cdot a_2 \otimes a_3, \quad a_1, a_2, a_3 \in J. \end{aligned}$$

We denote by J-mod the category of finite dimensional J-bimodules.

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1. Let A be an associative algebra we construct $A^+ = (A, \frac{1}{2}(ab+ba)), \quad A^- = (A, ab-ba)$.
2. For $\forall\ \mathfrak{g}$ $\dim_{\mathbf{k}} U(\mathfrak{g}) = \infty$ while $\dim_{\mathbf{k}} J < \infty \Rightarrow \dim_{\mathbf{k}} U(J) < \infty$.
3. For any Lie algebra $\mathfrak{g}$ its universal enveloping algebra $U(\mathfrak{g}) = F(\mathfrak{g})/\langle ab - ba - [a,b]\rangle$ and by PBW thm
$$\mathfrak{g} \subset U(\mathfrak{g})^- \quad \text{e} \quad \mathfrak{g}\text{-mod} \simeq U(\mathfrak{g})\text{-mod}.$$
4. $S(J) = F(J)/\langle ab + ba - 2a\cdot b\rangle$ is called **universal associative enveloping** of J and
$$J \subset S(J)^+ \iff J \text{ is special}, \quad J\text{-mod}_{\frac{1}{2}} \simeq S(J)\text{-mod}.$$
Let $\sigma : J \to \mathrm{End}_{\mathbf{k}} M$, satisfying for all $a, b \in J$
$$\sigma(a\cdot b) = \frac{1}{2}(\sigma(a)\sigma(b) + \sigma(b)\sigma(a)). \tag{3}$$
This gives the structure of J-bimodule on M via $\frac{1}{2}\sigma : J \to \mathrm{End}_{\mathbf{k}} M$, and is called **one-sided** (bi)modules.

Jordan algebras versus Lie algebras

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Let e be an identity element in J then M is called **unital** if $\rho(e)m = m$ for all $m \in M$, denote by J-mod$_1 \subset J$-mod, one can introduce the corresponding enveloping algebra $U_1(J)$.

Let $\rho : a \to \rho_a$ is a representation of J corresponding to M then

$$\rho_e(\rho_e - 1)(2\rho_e - 1) = 0$$

gives the Peirce decomposition for M

$$M = M_0 \oplus M_{\frac{1}{2}} \oplus M_1,$$

where $M_i = \{m_i \,|\, \rho_e m_i = i m_i\}$.

In particular $\rho : J \to \mathrm{End}_{\mathbf{k}} M_1$, is a unital representation, and $\rho : J \to \mathrm{End}_{\mathbf{k}} M_{\frac{1}{2}}$ is one-sided representation of J, while $\rho : J \to \mathrm{End}_{\mathbf{k}} M_0$ is one-dimensional trivial representation.

Therefore to describe J-mod it suffices to describe $S(J)$-mod and $U_1(J)$-mod.

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We start we the Albert classification of simple finite-dimensional:

- k;
- $J(V, f)$, f non-degenerate;
- $H_n(C)$, $n \geq 3$, (C, τ) composition algebra of dimension 1, 2, 4;
- $\mathcal{A}$, the Albert algebra.

In 1954, N. Jacobson described all irreducible modules for J simple:

J	$S(J)$	$U_1(J)$
$Sym_n(\mathbf{k})$	M_n	$M_{\frac{n(n+1)}{2}} \oplus M_{\frac{n(n-1)}{2}}$
$M_n(\mathbf{k})^+$	$M_n \oplus M_n$	$M_{n^2} \oplus M_{\frac{n(n+1)}{2}} \oplus M_{\frac{n(n+1)}{2}}$ $\oplus M_{\frac{n(n-1)}{2}} \oplus M_{\frac{n(n-1)}{2}}$
$Symp_{2n}(\mathbf{k})$	M_{2n}	$M_{n(2n-1)} \oplus M_{n(2n+1)}$ *
$J_n(V, f)$ dim $V = n$ is even	M_{2^n}	$\oplus_s M_s$ $s = \binom{n+1}{1}, \binom{n+1}{3}, \ldots, \binom{n+1}{n+1}$
$J_n(V, f)$ $n = 2\nu - 1$	$M_{2^{n-1}} + M_{2^{n-1}}$	$M_{\frac{1}{2}\binom{n+1}{\nu}} \oplus M_{\frac{1}{2}\binom{n+1}{\nu}} \oplus_s M_s$ $s = \binom{n+1}{0}, \binom{n+1}{1}, \ldots, \binom{n+1}{\nu-1}$
$\mathcal{A}$	0	$M_{27}(\mathbf{k})$

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$J_n(V,f)$ dim $V = n$ is even	M_{2^n}	$\oplus_s M_s$ $s = \binom{n+1}{1}, \binom{n+1}{3}, \dots, \binom{n+1}{n+1}$
$J_n(V,f)$ $n = 2\nu - 1$	$M_{2^{n-1}} + M_{2^{n-1}}$	$M_{\frac{1}{2}\binom{n+1}{\nu}} \oplus M_{\frac{1}{2}\binom{n+1}{\nu}} \oplus_s M_s$ $s = \binom{n+1}{0}, \binom{n+1}{1}, \dots, \binom{n+1}{\nu-1}$
$\mathcal{A}$	0	$M_{27}(\mathbf{k})$

We start we the Albert classification of simple finite-dimensional:

- $\mathbf{k}$;
- $J(V, f)$, f non-degenerate;
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In 1954, N. Jacobson described all irreducible modules for J simple:

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Let S_1 and S_2 be in J-mod$_{\frac{1}{2}}$ then we may form the **Kronecker sum** of two given one-sided modules $S = S_1 \otimes S_2 \in J$-mod$_1$ by setting

$$a(s_1 \otimes s_2) = as_1 \otimes s_2 + s_1 \otimes as_2.$$

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Representation type of algebra

Let **k** be algebraically closed field and A be an associative finite dimensional **k**-algebra. Then A is of

- a **finite representation type** if there are finitely many isomorphism classes of finitely generated, indecomposable left A-modules.
- a **tame representation type** if isomorphism classes of finitely generated, indecomposable left A-modules form in each dimension finitely many one-parameter families.
- or a **wild representation type**

$A \Longleftrightarrow$ quiver with relations $(Q(A), R)$ such that

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Inspired by results of P. Gabriel for associative case:

$$J = J_s + \mathrm{Rad} J, \qquad \mathrm{Rad}^2 J = 0$$

Let Q be a quiver, the **quiver double** $D(Q)$ of Q is defined as follows:

$$D(Q_0) = \{X^+, X^- \mid X \in Q_0\}$$
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Theorem (Gabriel)

Let A be a finite dimensional associative algebra over algebraically closed field, such that $\mathrm{Rad}^2 A = 0$, Q its quiver. Then A is of finite (tame) representation type if and only if $D(Q)$ is a disjoint union of simply-laced Dynkin diagrams (extended Dynkin diagrams).

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1. describe **matrix** Jordan algebras which are tame/finite with respect to their **one-sided** representation type: *K., Ovsienko S., Shestakov I., Representation type of Jordan algebras, Advances in Math., 2011.*
2. describe Jordan algebras of **Clifford type** which are tame/finite with respect to their **one-sided** representation type;
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The Tits-Kantor-Koecher construction

A **short grading** of $\mathfrak{g}$ is a $\mathbb{Z}$-grading of the form $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$.
Let P be the commutative bilinear map on J: $P(x, y) = x \cdot y$.
We associate to J a Lie algebra with short grading

$$Lie(J) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1.$$

Put $\mathfrak{g}_{-1} = J$, $\mathfrak{g}_0 = \langle L_a, [L_a, L_b] | a, b \in J \rangle$, $\mathfrak{g}_1 = \langle P, [L_a, P] | a \in J \rangle$

- $[L, x] = L(x)$ for $x \in \mathfrak{g}_{-1}$, $L \in \mathfrak{g}_0$;
- $[B, x](y) = B(x, y)$ for $B \in \mathfrak{g}_1$ and $x, y \in \mathfrak{g}_{-1}$;
- $[L, B](x, y) = L(B(x, y)) - B(L(x), y) + B(x, L(y))$ for any $B \in \mathfrak{g}_1$, $L \in \mathfrak{g}_0$ and $x, y \in \mathfrak{g}_{-1}$.

Then $\mathfrak{g} = Lie(J)$ is Lie algebra and is called the **Tits-Kantor-Koecher (TKK) construction** for J.

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A **short subalgebra** of $\mathfrak{g}$ is an $\mathfrak{sl}_2$ subalgebra spanned by e, h, f such that the eigenspace decomposition of *ad h* defines a short grading on $\mathfrak{g}$.

For any J with identity e consider in $Lie(J)$

$$h_J = -L_e, \quad f_J = P, \quad \text{then } \alpha_J = \langle e, h_J, f_J \rangle$$

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Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be the $\mathbb{Z}_2$-graded Lie algebra, $p \in \mathfrak{g}_1$. For any $x, y \in \mathfrak{g}_{-1}$ set

$$x \cdot y = [[p, x], y]$$

then $Jor(\mathfrak{g}) := (\mathfrak{g}_{-1}, \cdot)$ is a Jordan algebra.

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Relations between J-mod and $\mathfrak{g} = TKK(J)$-modules?

We define two adjoint functors *Jor* and *Lie* between J-mod and $\mathfrak{g}$-modules admitting a short grading.

Not every J-module can be obtained from a $\mathfrak{g}$-module by application of *Jor*: one has to consider $\hat{\mathfrak{g}}$ the universal central extension of $\mathfrak{g}$.

Let $\mathcal{S}$ (resp. $\mathcal{S}_{\frac{1}{2}}$) be the category of $\hat{\mathfrak{g}}$-modules M such that the action of α_J induces a short grading on M (resp. a grading of length 2, namely $M_{-\frac{1}{2}} \oplus M_{\frac{1}{2}}$).

To define *Jor* let $N \in \mathcal{S}$. Then $N = N_1 \oplus N_0 \oplus N_{-1}$. We set $Jor(N) := N_{-1}$

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It is clear that *Jor* is an exact functor.

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Let $M \in J - mod_1$. Let $\mathcal{A} = Lie(\mathcal{J} \oplus M)$. Then we have an exact sequence

$$0 \to N \to \mathcal{A} \xrightarrow{\pi} \mathfrak{g} \to 0, \tag{4}$$

where N is an abelian Lie algebra and $N_{-1} = M$.
N is a $\hat{\mathfrak{g}}$-module, thus $N_{-1} = M$ is $\hat{\mathfrak{g}}_0$-module.

Let $\mathcal{P} = \hat{\mathfrak{g}}_0 \oplus \mathfrak{g}_{-1}$ and we extend the above $\hat{\mathfrak{g}}_0$-module structure on M to a $\mathcal{P}$-module structure by setting $\mathfrak{g}_{-1} M = 0$. Next

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- $Jor \circ Lie$ is isomorphic to the identity functor in $J - mod_1$.
- Let $N \in \mathcal{S}$ and $\hat{\mathfrak{g}}N = N$, then the canonical map $Lie(Jor(N)) \to N$ is surjective.
- Let $N \in \mathcal{S}$ and $N^{\hat{\mathfrak{g}}} := \{x \in N \,|\, \hat{\mathfrak{g}}x = x\} = 0$, then the canonical map $N \to Lie(Jor(N))$ is injective.
- If $M \to L \to 0$ is exact in $J - mod_1$, then $Lie(M) \to Lie(L) \to 0$ is exact in $\mathcal{S}$.

The splitting $J - mod_1 \oplus J - mod_0$ can not be lifted to the Lie algebra $\hat{\mathfrak{g}}$, since some modules can have non-trivial extensions with trivial modules, thus left and right adjoint of the functor *Jor* are not isomorphic and the categories $\mathcal{S}$ and $J - mod_1$ are not equivalent. Still they are close enough and one can describe projective modules, quivers and relations of $J - mod_1$ in terms of $\mathcal{S}$.

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Representation table for $TKK(J)$

J	$\mathfrak{g}$	$\mathcal{S}_{\frac{1}{2}}$	$\mathcal{S}$
$Sym_n(\mathbf{k})$	$\mathfrak{sp}_{2n}$	V	$ad,\ \Lambda^2 V$
$M_n(\mathbf{k})^+$	$\mathfrak{sl}_{2n}$	$V,\ V^*$	$ad,\ S^2(V),\ S^2(V^*),\ \Lambda^2(V),\ \Lambda^2(V^*)$
$Symp_{2n}(\mathbf{k})$	$\mathfrak{so}_{4n}$	V	$ad,\ S^2(V)$
$J_n(E,f)$ $n=2\nu$	$\mathfrak{so}_{n+3}$	Γ spinor	$\Lambda^i(V),\ i=1,\ldots,\nu+1$
$J_n(E,f)$ $n=2\nu-1$	$\mathfrak{so}_{n+3}$	$\Gamma^+,\ \Gamma^-$ spinor	$\Lambda^i(V),\ i=1,\ldots,\nu$ $\Lambda^{\nu+1}(V)^{\pm}$
$\mathcal{A}$	E_7		ad

Let $\mathcal{C}$ be an abelian category with finitely many simple modules such that every object has finite length and every simple object has a projective cover.
Then $\mathcal{C}$ is equivalent to the category of finite-dimensional A-modules. If $L_1, \dots, L_r$ is the set of all up to isomorphism simple objects in $\mathcal{C}$ and $P_1, \dots, P_r$ are their projective covers, then A is a pointed algebra which is usually realized as the path algebra of a certain quiver Q with relations.
The vertices

$$Q_0 = \{\text{simple modules } L_1, \dots, L_r\}$$

$$Q_1 = \{\#\text{arrows from vertex } L_i \text{ to vertex } L_j \text{ is } \dim \mathrm{Ext}^1(L_j, L_i)\}$$

It is now clear how to describe the quiver Q.

Lemma

Let $\mathfrak{g} = \mathfrak{g}_s + \mathfrak{r}$ be the Levi decomposition of $\mathfrak{g}$. Let L and L' be two simple $\mathfrak{g}_s$-modules then $\dim \operatorname{Ext}^1(L, L')$ equals the multiplicity of L' in $L \otimes \mathfrak{r}$.

Example

Let $J = M_n{}^{(+)}(\mathbf{k}) + Sym_n$ then its $TKK(J) = sl_{2n} + S^2(V)$, while $Q_{\mathcal{S}_{\frac{1}{2}}}(J)$ and $Q_{\mathcal{S}}(J)$ are correspondingly

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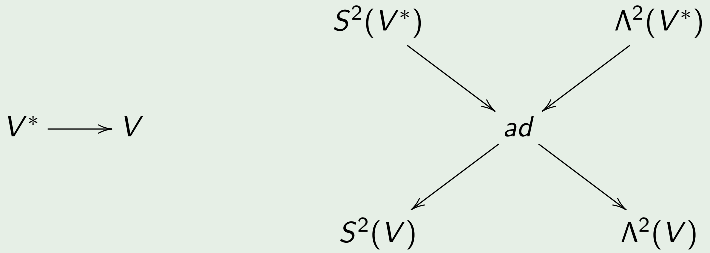
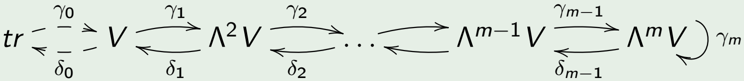


Example

$\mathfrak{g} = \mathfrak{so}_{2m+1} \oplus V$, $m \geq 3$



with the relations

$$\gamma_{r-1}\gamma_r = \delta_r\delta_{r-1} = 0,$$
$$\gamma_{r-1}\delta_{r-1} = \delta_r\gamma_r,$$
$$\gamma_{m-1}\delta_{m-1} = \gamma_m^2,$$
$$\text{for } r = 1, \ldots, m-1.$$