Structure and representations of Jordan superalgebras

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Pascual Jordan, 1933:

Definition

Jordan algebra is an algebra over a field of characteristic not 2 that satisfies the identities

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\begin{aligned}
xy &= yx, \\
(x^2y)x &= x^2(yx).
\end{aligned}
$$

A associative with a product $ab \Longrightarrow A^{(+)} = \langle A, +, \cdot \rangle$ is Jordan, where $a \cdot b = \frac{1}{2}(ab + ba)$.

A Jordan algebra J is called *special* if there exists an associative algebra A such that $J \subseteq A^{(+)}$.

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Examples:

Example (Algebras of Hermitian type)

$(A, *)$ associative algebra with involution $*$,
$H(A, *) = \{a \in A \,|\, a^* = a\}$ is a (Jordan) subalgebra of $A^{(+)}$.

$(A, *)$ is $*$-simple $\Longrightarrow H(A, *)$ is simple.

Example (Algebras of Clifford type (spin-factors))

V a vector space over a field F, $f : V \times V \to F$ a symmetric bilinear form, $J = F \cdot 1 \oplus V$ such that $1_J = 1$, $u \cdot v = f(u, v)1$ for $u, v \in V$. Then $J = J(V, f)$ is a Jordan algebra. If $\dim V > 1$ and f is non-degenerate, then $J(V, F)$ is simple

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Example (Algebras of Albert type)

$(\mathbb{O}, *)$ an algebra of (generalized) octonions with the standard involution, $H(\mathbb{O}_3, *)$ the space of $*$-hermitian 3×3 matrices over $\mathbb{O}$. Then $H(\mathbb{O}_3, *)$ is a simple non-special Jordan algebra with respect to multiplication $a \cdot b = \frac{1}{2}(ab + ba)$.

A Jordan algebra J is *of Albert type* if $\overline{J} = \bar{F} \otimes J \cong H(\mathbb{O}_3, *)$ where $\bar{F}$ is the algebraic closure of the field F.

Theorem (E.Zelmanov)

Every simple Jordan algebra is of Hermitian, Clifford, or Albert type.

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Every simple Jordan algebra is of Hermitian, Clifford, or Albert type.

A *superalgebra*, in general, is a Z_2-graded algebra,

$$A = A_0 \oplus A_1, \ A_i A_j \subseteq A_{i+j(mod\,2)}.$$

Examples

- $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$, $\mathbb{C}_0 = \mathbb{R}$, $\mathbb{C}_1 = \mathbb{R}i$;
- A an algebra $\Longrightarrow A[\sqrt{1}] = A \oplus Au$ (A-double), $(A[\sqrt{1}])_0 = A, (A[\sqrt{1}])_1 = Au$, $u(=\sqrt{1})$ central with $u^2 = 1$;
- Grassmann algebra $G = alg\,\langle 1, e_1, e_2, \cdots \mid e_i e_j = -e_j e_i \rangle$, $G = G_0 \oplus G_1$.

Grassmann envelope $G(A) = G_0 \otimes A_0 + G_1 \oplus A_1$.

A superalgebra $A = A_0 + A_1$ is a $\mathcal{M}$*-superalgebra* if $G(A) \in \mathcal{M}$ ($\mathcal{M} = Assoc, Lie, Jord$, etc).

$A \in \mathcal{M} \Longrightarrow G \otimes A = (G_0 \otimes A) \oplus (G_1 \otimes A)$ is a $\mathcal{M}$-superalgebra.

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Associative superalgebras

Theorem (C.T.C.Wall, 1963)

Every simple finite-dimensional associative superalgebra over an algebraically closed field F is isomorphic to one of the following superalgebras:

- $A = \mathrm{M}_{m|n}(F), \quad A_{\bar 0} = \left\{ \begin{pmatrix} \star & 0 \\ 0 & \star \end{pmatrix} \begin{matrix} m \\ n \end{matrix} \right\}, \quad A_{\bar 1} = \left\{ \begin{pmatrix} 0 & \star \\ \star & 0 \end{pmatrix} \begin{matrix} m \\ n \end{matrix} \right\},$
- $A = \mathrm{M}_n(F)[\sqrt{1}\,],$ *the doubled matrix algebra.*

In general, if $A \in \mathcal{M}$ then the A-double $A[\sqrt{1}]$ is not an $\mathcal{M}$-superalgebra.

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Jordan superalgebras

$A = A_0 \oplus A_1$ associative superalgebra with a product $ab \Longrightarrow$ $A^{(+)} = \langle A, +, \cdot \rangle$ is a Jordan superalgebra , where $a \cdot b = \frac{1}{2}(ab + (-1)^{|a||b|}ba),\ |a| = i$ if $a \in A_i$.

Example

$M_n(F)[\sqrt{1}]^{(+)}\ (n > 1),\ M_{m|n}(F)^{(+)}$ are simple Jordan superalgebras.

A linear mapping $* : A_0 \oplus A_1 \to A_0 \oplus A_1$ is called *a superinvolution* if $(ab)^* = (-1)^{|a||b|} b^* a^*,\ (a^*)^* = a$.

Example (Hermitian superalgebras)

$(A, *)$ associative superalgebra with superinvolution $*$, $H(A, *) = \{a \in A \,|\, a^* = a\}$ is a (Jordan) subsuperalgebra of $A^{(+)}$.

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$M_{n|n}(F)$ has a superinvolution (supertransposition)

$$trp : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} D^t & -B^t \\ C^t & A^t \end{pmatrix}$$

$M_{n|2m}(F)$ has an ortho-symplectic superinvolution

$$osp : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} I_n & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} A^t & -C^t \\ B^t & D^t \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & -U \end{pmatrix},$$

where I_n is the identity matrix of order n and $U = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}$.

Examples

Hermitian simple Jordan superalgebras
$Jtrp(n) = H(M_{n|n}, trp), \ (n > 1), \ Josp(n, 2m) = H(M_{n|2m}, osp)$.

Example (Superalgera of superform)

$V = V_0 \oplus V_1$ a vector superspace over a field F, $f : V \times V \to F$ a supersymmetric bilinear form, that is, $f|_{V_0}$ is symmetric, $f|_{V_1}$ is skew-symmetric, $f(V_i, V_j) = 0$ if $i \neq j$. Then $J(V, f) = F \cdot 1 \oplus V$ such that $1_J = 1, \ u \cdot v = f(u, v)1$ for $u, v \in V$ is a Jordan superalgebra, $J_0 = F \cdot 1 \oplus V_0, \ J_1 = V_1$. If f is non-degenerate then $J(V, f)$ is simple, except the case dim $V = 1, \ V = V_0$.

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Example (Kaplansky superalgebra K_3)

$K_3 = Fe + (Fx + Fy)$, where $e^2 = e$, $ex = \frac{1}{2}x$, $ey = \frac{1}{2}y$, $xy = e$. The superalgebra K_3 is simple not unital.

Example (Superalgebra D_t)

$D_t = (Fe_1 + Fe_2) + (Fx + Fy)$ with the product: $e_i^2 = e_i$, $e_1 e_2 = 0$, $e_i x = \frac{1}{2}x$, $e_i y = \frac{1}{2}y$, $xy = e_1 + te_2$, $t \in F$, $i = 1, 2$. The superalgebra D_t is simple if $t \neq 0$. In the case $t = -1$, the superalgebra D_{-1} is isomorphic to $M_{1|1}(F)^{(+)}$.

Example (Kac superalgebra K_{10})

V. Kac introduced the simple 10-dimensional superalgebra K_{10} that is related (via the Tits-Kantor-Koecher construction) to the exceptional 40-dimensional Lie superalgebra.

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Kantor double

Example (Kantor double)

Let $A = A_0 \oplus A_1$ be an associative commutative superalgebra equipped with a graded super-anticommutative bilinear map (bracket)

$$[,] : A \otimes A \to A, \ [A_i, A_j] \subseteq A_{i+j}.$$

A *Kantor double* is a direct sum of vector spaces $J(A, [,]) = A \oplus \bar{A}$, with the product

$$a \cdot b = ab, \ a \cdot \bar{b} = \overline{ab}, \ \bar{b} \cdot a = (-1)^{|a|}\overline{ba}, \ \bar{a} \cdot \bar{b} = (-1)^{|b|}[a, b].$$

$J = J(A, [,])$ is a commutative superalgebra under the grading $J_0 = A_0 \oplus \bar{A}_1, \ J_1 = A_1 \oplus \bar{A}_0$. A bracket $[,]$ is called a *Jordan bracket* on A if $J(A, [,])$ is a Jordan superalgebra.

I.Kantor showed that every Poisson bracket is Jordan. In particular, let V be an n-dimensional vector space with a basis $e_1, \ldots, e_n,\ n \geq 2$. Consider the Poisson bracket on the Grassmann superalgebra $G_n = G(V)$:

$$[f, g] = \sum_{i=1}^{n} (-1)^{|g|} \frac{\partial f}{\partial e_i} \frac{\partial g}{\partial e_i}.$$

The Jordan superalgebra $Kan(n) = J(G(V), [,])$ is simple, $\dim(Kan(n)) = 2^{n+1}$.

Theorem (V.Kac, with a rectification of I.Kantor)

Every finite dimensional simple Jordan superalgebra over an algebraically closed field of characteristic 0 is isomorphic to one of the superalgebras

$$M_n[\sqrt{1}]^{(+)},\ M_{n|m}^{(+)},\ Jtrp(n), (n > 1),\ Josp(n, 2m),\ J(V, f)$$
$$K_3,\ D_t,\ K_{10},\ Kan(n).$$

Finite dimensional simple Jordan superalgebras in positive characteristic were classified in papers by E.Zelmanov, C.Martínez, and M.Racine. In this case appear some new superalgebras, namely, the *Cheng-Kac superalgebra JCK_6*, Kantor doubles $J(A, [,])$ where the Jordan bracket [,] is not necessary Poisson, and exceptional matrix superalgebras in characteristic 3.

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S.Eilenberg, 1947:

Let A be an algebra and V be an A-bimodule. The *split null extension* $E(A, V) = A \oplus V$ is an algebra where A is a subalgebra, $V^2 = 0$, and the products $a \cdot v$, $v \cdot a$ for $a \in A$ and $v \in V$ are given by the bimodule action.

If $A \in \mathcal{M}$ ($\mathcal{M} = Assoc, Lie, Jord$, etc) then the bimodule V is called an *$A_{\mathcal{M}}$-bimodule* if $E(A, V) \in \mathcal{M}$.

The *universal multiplicative enveloping algebra* $U_{\mathcal{M}}(A)$ is an associative algebra such that $A_{\mathcal{M}}$-bimod $\cong U_{\mathcal{M}}(A)$-mod.

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N.Jacobson:

$U(J) = U_{Jord}(J)$.

- $\dim J < \infty \Longrightarrow \dim U(J) < \infty$;
- J semisimple $\Longleftrightarrow U(J)$ is semisimple $\Longleftrightarrow$ every bimodule is comletely reducible;
- $\dim J < \infty \Longrightarrow J$ has a finite number of irreducible bimodules;
- $J^n = 0 \Longrightarrow J$ has no irreducible bimodules;
- Finite dimensional irreducible bimodules were classified.

Coordinatization Theorem

L.Kronecker: A associative, $A \supseteq M_n(F) \ni 1_A \Longrightarrow A \cong M_n(B)$
A-bimod $\cong$ B-bimod.

N.Jacobson: J Jordan, $J \supseteq H_n(F) \ni 1_J \Longrightarrow J \cong H_n(D)$, where D is an alternative algebra with a nuclear involution.

$H_n(D)_{Jord}$-bimod $\cong (D,*)_{Alt}$-bimod.

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Jordan supermodules

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Difference: a nilpotent Jordan superalgebra may have irreducible supermodule.

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Jordan supermodules

- In *char* $F = 0$ case, the TKK (Tits-Kantor-Koecher) functor admits to apply results from representation theory of Lie superalgebras.
- For superalgebras of classical type (Hermitian and Clifford superalgebras) sometimes the classical methods work in positive characteristic as well (coordinatization theorems, etc.).
- The most difficult case: non-classical superalgebras in positive characteristic.

Char 0 case

A.Stern: K_{10}, $Kan(n), (n > 4)$ are *rigid*, that is, have only regular irreducile supermodules. ($Kan(n)$- ???)

C.Martínez and E.Zelmanov: All irreducible bimodules over finite dimensional simple Jordan superalgebras over an algebraically closed field of characteristic 0 were classified.

Matrix Jordan superalgebras of big order ($n > 2$) and the superalgebra $J(V, f)$ behave similar to the non-graded case and are of *finite type*, that is, have a finite number of indecomposable supermodules, and each of them is irreducible. The same is true for the superalgebra $M_2[\sqrt{1}]^{(+)}$.

The superalgebras K_3, D_t, $M_{1|1}^{(+)}$, $Jtrp(2)$ have infinite number of irreducible supermodules.

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M.C. López Díaz and I.Sh.;
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Coordinatization theorem for Jordan superalgebras:

Let J be a unital Jordan superalgebra such that $J_0 \supseteq H_n(F) \ni 1_J$, $n > 2$. Then there exists an alternative superalgebra with a nuclear superinvolution $(D, *)$ such that $J \cong H_n(D, *)$.

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Irreducible bimodules over $M_n[\sqrt{1}]^{(+)}$, $Jtrp(n)$ are classified for $n > 2$ in characteristic $p > 2$.

Open question: Bimodules over $Josp(n, 2m)$, $M_{n|m}^{(+)}$ in positive characteristic.

Characteristic $p > 2$: coordinatization theorem

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Superalgebra $Kan(n)$, $char$ 0 case, $n > 4$

We have $Kan(n) = J(G_n, [,])$, where G_n is the Grassman algebra on anticommuting variables $e_1, \ldots, e_n$ with the bracket $[f, g] = \sum_{i=1}^{n} (-1)^{|g|} \frac{\partial f}{\partial e_i} \frac{\partial g}{\partial e_i}$.

Let $A = G_n[t]$ be the algebra of polynomials over G_n. Extend the bracket on A by setting $[t, e_i] = 0$, $[t, 1] = \alpha t$, $\alpha \in F$.

In the Kantor double $J(A, [,]) = A \oplus \bar{A}$, the subsuperalgebra $G_n \oplus \overline{G_n}$ is isomorphic to $Kan(n)$, whereas the subspace $V(\alpha) = tG_n \oplus \overline{tG_n}$ is an irreducible unital bimodule over it.

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O.Folleco Solarte and I.Sh.:

Every irreducible finite dimensional Jordan $Kan(n)$-bimodule for $n \geq 2$ and $char\ F \neq 2$ is (up to change of parity) isomorphic to $V(\alpha)$.

$V(\alpha)$ is Jordan

A linear operator E on a unital superalgebra A is called a *generalized derivation* of A if $E(ab) = E(a)b + aE(b) - abE(1)$. Let $P = \langle P_0 \oplus P_1, \{,\}\rangle$ be a unital Poisson superalgebra, $E : P \to P$ be a generalized derivation of P which satisfies also the condition $E(\{p, g\}) = \{E(p), q\} + \{p, E(q)\} + \{p, q\}E(1)$. Furthermore, let (A, D) be a commutative associative algebra with a derivation D. Define the following bracket on the tensor product $P \otimes A$:

$$\langle p\otimes a, q\otimes b\rangle = \{p, q\}\otimes ab + E(p)q\otimes aD(b) - (-1)^{|p||q|}E(q)p\otimes D(a)b$$

where $p, q \in P$; $a, b \in A$.

The defined bracket is a Jordan bracket on the commutative and associative superalgebra $P \otimes A = (P_0 \otimes A) \oplus (P_1 \otimes A)$.

$V(\alpha)$ is Jordan

Define a linear operator E on the Poisson superalgebra G_n by setting $E(e_{i_1}\cdots e_{i_k}) = (k-1)e_{i_1}\cdots e_{i_k}$, then E is a generalized derivation of G_n that satisfied the above conditions on the bracket $\{,\}$. Let $A = F[t]$ and $D_\alpha = -\alpha t\frac{d}{dt} \in Der\, A$, then the superalgebra $(G_n, [,]) \otimes (A, D_\alpha)$ with the bracket defined as above is isomorphic to $(G_n[t], [,])$. Therefore, the bracket $[,]$ on $G[t]$ is Jordan, and the supermodule $V(\alpha)$ is Jordan.