

# Some open conjectures on linear codes from finite planes

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## Abstract

We examine some outstanding open questions regarding the code from the row span over a finite field of an incidence matrix of a finite plane. In particular, we show that:

- The Hall planes of even order provide an infinite class of finite non-desarguesian projective planes that satisfy the Hamada-Sachar conjecture that the desarguesian planes have the smallest dimension for planes of a given order. The planes and their duals are not tame. (Joint work with McDonough and Mavron[KMM14].)
- There are non-desarguesian affine planes of order 16 that have words the weight of an affine line but that are not the incident vectors of affine lines. (Joint work with Ghinelli and de Resmini[GdRK08].)

Codes from the row span over a finite field of an incidence matrix of a finite projective plane have an important historical role in that the combinatorial structure of the plane led to the usefulness of their dual codes through the use of majority logic decoding, and the subsequent coincidence of interests of coding theorists and finite geometers in the 1960s.

This coincidence of interests continues to the present day.

For the finite geometers, the codes could be used for classification of planes.

We discuss progress in this classification here.

- A  $t$ - $(v, k, \lambda)$  design is an incidence structure  $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ , with point set  $\mathcal{P}$ , block set  $\mathcal{B}$  and incidence  $\mathcal{I}$  such that  $|\mathcal{P}| = v$ , every block  $B \in \mathcal{B}$  is incident with precisely  $k$  points, and every  $t$  distinct points are together incident with precisely  $\lambda$  blocks.
- A  $2$ - $(n^2 + n + 1, n + 1, 1)$  design, for  $n \geq 2$ , is a **projective plane of order  $n$** .
- A  $2$ - $(n^2, n, 1)$  design obtained by removing a line and all the points on it, from a projective plane is an **affine plane of order  $n$** .
- $PG_2(\mathbb{F}_q)$  is the **desarguesian** projective plane, i.e. the design of points and lines of projective 2-space over  $\mathbb{F}_q$ ;  $AG_2(\mathbb{F}_q)$  is the **desarguesian** affine plane.

## More terminology

- The **code**  $C_F(\mathcal{D})$  of the design  $\mathcal{D}$  over  $F$  is the space spanned by the incidence vectors of the blocks over  $F$ ,

$$C_F(\mathcal{D}) = \langle v^B \mid B \in \mathcal{B} \rangle$$

where  $v^Q$  is the incidence vector of a subset  $Q$  of points.

- Equivalently, this is the row span over  $F$  of a **blocks by points incidence matrix** for  $\mathcal{D}$ .
- For planes of order  $n$  with  $p$  prime and  $p|n$ , take  $F = \mathbb{F}_p$ .
- The **hull** of a code  $C$  or design  $\mathcal{D}$  with  $C = C_F(\mathcal{D})$ , is

$$\text{Hull}(C) = C \cap C^\perp$$

## Well-known facts

For  $\Pi$  any projective plane of order  $n$ ,  $p \mid n$ ,  $p$  a prime,

- $C = C_p(\Pi)$  has **minimum weight**  $n + 1$ ;
- the **minimum words** of  $C$  are the scalar multiples of the incidence vectors of the lines;
- $\text{Hull}_p(\Pi) = C \cap C^\perp = \langle v^L - v^M \mid L \text{ and } M \text{ lines of } \Pi \rangle$ .

(easy to prove [AK92, Theorem 6.3.1]).

For **desarguesian planes**  $\Pi = PG_2(\mathbb{F}_{p^t})$ , if  $C = C_p(\Pi)$ , then

- $\dim(C) = \binom{p+1}{2}^t + 1$ ;
- $\text{Hull}(C)$  has **minimum weight**  $2p^t$ ;
- the **minimum words** of  $\text{Hull}(C)$  are  $\alpha(v^L - v^M)$ , for  $L, M$  lines of  $\Pi$ ,  $\alpha \in \mathbb{F}_p^*$ .

(from Delsarte, Goethals and MacWilliams, see [AK92, Chapters 5,6]).

The notion of a **tame plane** was introduced in [AK92, Section 6.9]:

## Definition

A projective plane  $\Pi$  of order  $n$  is said to be **tame** (or tame at  $p$ , where  $p|n$  is a prime) if

- $\text{Hull}_p(\Pi)$  has minimum weight  $2n$ ;
- the minimum-weight vectors of  $\text{Hull}_p(\Pi)$  are precisely the scalar multiples of the differences of the incidence vectors of two distinct lines of  $\Pi$ .

- The **desarguesian planes** are the only ones known to be **tame**; many non-desarguesian planes of small order have been shown not to be tame, either because the minimum weight of the hull is not  $2n$  (see [GdRK08]) or, more frequently, that there are words of weight  $2n$  that are not scalar multiples of the differences of the incidence vectors of two lines.
- We show here that the Hall planes of even order  $2^{2t}$  for  $t \geq 2$  are not tame by exhibiting words of weight  $2^{2t+1}$  in the binary hull that are not differences of the incidence vectors of two lines. From [KdR98, Corollary 3], this shows that the **even order Hall planes and their dual planes are not tame** for all even orders  $n > 4$ .



# Hamada-Sachar conjecture

- An outstanding conjecture concerning codes from projective planes is the **Hamada-Sachar conjecture** [AK92, Conjecture 6.9.1]:

## Conjecture

*Every projective plane of order  $p^s$ ,  $p$  a prime, has  $p$ -rank at least  $\binom{p+1}{2}^s + 1$  with equality if and only if it is desarguesian.*

- This has been demonstrated computationally for many individual planes of small order.
- We show here that the Hall planes of even order  $q = 2^t$  have binary codes with **dimension greater than that of the desarguesian plane**, i.e. greater than  $3^t + 1$ , thus reaffirming the conjecture for an infinite class of planes.

# Baer segments and hyperovals

- If  $\Pi$  is a projective plane of square order  $n^2$ , a subplane  $\pi$  of  $\Pi$  of order  $n$  is called a **Baer subplane**. Lines of  $\Pi$  meet  $\pi$  in 1 or  $(n + 1)$  points. If a line of  $\Pi$  meets  $\pi$  in a set  $\delta$  of  $n + 1$  points,  $\delta$  is called a **Baer segment**.
- A **hyperoval** in a projective plane of even order  $n$  is a set of  $n + 2$  points such that lines meet the set in 0 or two points.

From Key, McDonough and Mavron [KMM14]:

## Proposition

If  $q = 2^t$ ,  $t \geq 2$ ,  $\Pi = PG_2(\mathbb{F}_{q^2})$ ,  $\mathcal{H}$  the projective Hall plane of order  $q^2$ , then

- $\text{Hull}(C_2(\mathcal{H}))$  contains words of weight  $2q^2$  having support the symmetric difference of two Baer subplanes that intersect in a line;
- neither  $\mathcal{H}$  nor its dual plane  $\mathcal{H}'$  is tame;
- $\dim(C_2(\mathcal{H})) > \dim(C_2(\Pi)) = 3^{2t} + 1$ .

Thus the Hall planes of even order confirm the **Hamada-Sachar conjecture**.

# Construction of Hall planes

Define the affine Hall plane  $\mathcal{AH}$  and the projective Hall plane  $\mathcal{H}$  through derivation by Baer segments.

- $\Pi = PG_2(\mathbb{F}_{q^2})$ ,  $q = p^e$ ,  $p$  prime;
- $\mathcal{L}$  the lines of  $\Pi$ ,  $\ell_\infty \in \mathcal{L}$ ,  $\mathcal{A} = AG_2(\mathbb{F}_{q^2}) = \Pi^{\ell_\infty}$ ;
- $\delta$  is a Baer segment of  $\ell_\infty$ ,  $|\delta| = q + 1$ ;
- $\mathcal{B} = \{\pi \mid \pi \text{ a Baer subplane of } \Pi, \pi \supset \delta\}$ ,  $|\mathcal{B}| = q^2(q + 1)$ ;  
for  $\pi_1, \pi_2 \in \mathcal{B}$ ,  
 $\pi_1 \cap \pi_2 = \delta$  or  $\delta \cup \{R\}$ , some point  $R$  off  $\ell_\infty$ .
- For  $\pi \in \mathcal{B}$ ,  $\{\pi^* \in \mathcal{B} \mid (\pi \setminus \delta) \cap (\pi^* \setminus \delta) = \emptyset\} \cup \{\pi\}$   
form a set of  $q^2$  subplanes that will form a parallel class of lines in the new affine (Hall) plane  $\mathcal{AH}$ .
- There are  $q + 1$  of these parallel classes of subplanes.

# Construction of affine Hall plane $\mathcal{AH}$

- $\mathcal{L}_c = \{l \in \mathcal{L} \mid l \cap l_\infty \in l_\infty \setminus \delta\}$ , projective lines common to  $\Pi$  and  $\mathcal{H}$ ;  
 $|\mathcal{L}_c| = q^3(q-1)$ .
- $\mathcal{L}_o = \{l \in \mathcal{L} \mid l \cap l_\infty \in \delta\}$ .
- $\mathcal{L} = \mathcal{L}_c \cup \mathcal{L}_o \cup \{l_\infty\}$ , the lines of  $\Pi$ .
- $\mathcal{A}_c = \{m \setminus l_\infty \mid m \in \mathcal{L}_c\}$ , affine lines common to  $\mathcal{A}$  and  $\mathcal{AH}$ .
- $\mathcal{A}_n = \{\pi \setminus \delta \mid \pi \in \mathcal{B}\}$ , the remaining  $q^2(q+1)$  affine lines of  $\mathcal{AH}$ .
- $\mathcal{A}_c \cup \mathcal{A}_n$  the lines of the affine Hall plane  $\mathcal{AH}$ .

# Construction of projective Hall plane $\mathcal{H}$

- For the projective Hall plane  $\mathcal{H}$ , adjoin a line at infinity

$$\ell_{\infty}^h = \{X \mid X \in \ell_{\infty} \setminus \delta\} \cup \{X_i \mid 0 \leq i \leq q\}$$

where the  $X_i$  correspond to the parallel classes of members of  $\mathcal{B}$ .

- Lines of  $\mathcal{H}$  are  $\mathcal{L}_c$ ,  $\ell_{\infty}^h$  and

$$\mathcal{L}_n = \{(\pi \setminus \delta) \cup \{X_i\} \mid \pi \in \mathcal{B}\}$$

where  $X_i$  corresponds to the parallel class containing  $\pi$ .

- For  $m \in \mathcal{L}_o$ ,  $m \setminus \delta$  is an affine Baer subplane of  $\mathcal{H}$ ;  
if  $m_1, m_2 \in \mathcal{L}_o$  and  $m_1 \cap m_2 \in \delta$  these planes are disjoint;  
in  $\mathcal{H}$  they share points only on  $\ell_{\infty}^h$ .

# The proof for $q = 2^t$

Now take  $q = 2^t$ .

Lines of  $\Pi$ :  $\mathcal{L}_c \cup \mathcal{L}_o \cup \{\ell_\infty\}$ ; lines of  $\mathcal{H}$ :  $\mathcal{L}_c \cup \mathcal{L}_n \cup \{\ell_\infty^h\}$ .

$E = \langle v^\ell \mid \ell \in \mathcal{L}_c \rangle \subseteq C_p(\Pi) \cap C_p(\mathcal{H})$ .

Show that for any  $R \in \delta$ , and for any lines  $m_1, m_2 \in \mathcal{L}_o$ , and  $R \in m_1, m_2$ ,

$$v^{m_1} - v^{m_2} \in E. \quad (1)$$

This, and using the fact that  $\Pi$  is tame, proves that

- $\dim(E) = \dim(C_2(\Pi)) - (2^t + 1) = 3^{2t} - 2^t$ ;
- $\mathcal{H}$  is not tame if  $t \geq 2$ ;
- $\dim(C_2(\mathcal{H})) > \dim(C_2(\Pi)) = 3^{2t} + 1$ .

These are true for any  $q$  if Equation (1) holds.

- Computation with Magma[CSW06, BCP97] indicates that Equation (1) holds for **any** subset of the line of size  $q + 1$ ; in fact if  $q = p^t$  then the same appears to be true for any subset on the line of size  $p^{t-1} + 1$ .
- We have only proved it for  $q = 2^t$  and Baer segments, and the proof uses hyperovals in the dual plane. The analogue for odd  $q$  is a word in the dual code with certain properties: work in progress. We have such a word for  $q = 3$  and the Hall plane of order 9: the difference of two Baer subplanes meeting in a line and a point.
- If we can prove Equation (1) for any odd  $q^2$ , then all Hall planes will satisfy the Hamada-Sachar conjecture, and would not be tame.



## Lemma

Let  $\Pi = (\mathcal{P}, \mathcal{L})$  be a projective plane with a polarity  $\sigma$ .

Write  $X^\sigma = X'$  for  $X \in \mathcal{P} \cup \mathcal{L}$ .

Let  $p$  be a prime dividing the order of  $\Pi$ , and  $C = C_p(\Pi)$ .

$$\text{If } w = \sum_X a_X v^X \in C^\perp, \text{ then } w' = \sum_X a_X v^{X'} = 0.$$

**Proof:** For  $Y \in \mathcal{P}$ , the value of  $w'$  at  $Y$  is

$$w'(Y) = \sum_{Y \in X'} a_X = \sum_{X \in Y} a_X = (w, v^{Y'}) = 0$$

since  $w \in C^\perp$ . Thus  $w' = 0$ . ■

## Binary case

In  $\Pi = PG_2(q^2)$ ,  $q = 2^t$ ,  $\delta$  a Baer segment on  $\ell_\infty = P'$  (homogeneous coordinates,  $P$  a point), and  $R \in \delta$ :

- 1 find a hyperoval  $\mathcal{O} = \{P_i, Q_i \mid 0 \leq i \leq q^2/2\}$  in  $\Pi$  such  $P \notin \mathcal{O}$ , and for one secant  $\ell$  through  $P$ ,  $\ell' = R$ , but for every other secant  $m$  through  $P$ ,  $m' \notin \delta$ ;
- 2 since  $v^{\mathcal{O}} = \sum_i (v^{P_i} + v^{Q_i}) \in C_2(\Pi)^\perp$ , then

$$\sum_i (v^{P_i} + v^{Q_i}) = 0;$$

- 3 if  $\ell = P_0Q_0$  and the other secants through  $P$  are  $P_iQ_i$  for  $1 \leq i \leq \frac{q^2}{2}$  then

$$v^{P_0} + v^{Q_0} = \sum_{i=1}^{\frac{q^2}{2}} (v^{P_i} + v^{Q_i}), \quad (2)$$

and  $R = P'_0 \cap Q'_0$ , where  $P'_0, Q'_0 \in \mathcal{L}_o$ ,  $P'_i, Q'_i \in \mathcal{L}_c$  for  $i > 0$ .

## Finding $\delta$ and $\mathcal{O}$

For some  $z \in K = \mathbb{F}_{q^2}$  let the Baer segment be

$$\delta = \{(1, 1, t + z) \mid t \in \mathbb{F}_q\} \cup \{(0, 0, 1)\},$$

let  $R = (0, 0, 1)$ ,  $P = (1, 1, 0)$ ,  $\ell_\infty = (1, 1, 0)'$ ,

$$\mathcal{O} = \{(1, y, y^{-1}) \mid y \in K^\times\} \cup \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}.$$

$H = \{y^2 + y \mid y \in K\}$  is an additive subgroup of  $K$ ,  $|H| = \frac{q^2}{2}$ ,  $H > \mathbb{F}_q$ .

If we choose  $z \notin H$  then the lines  $\{(1, 1, t + z)' \mid t \in \mathbb{F}_q\}$  are exterior to  $\mathcal{O}$ .

## Secants through $P = (1, 1, 0)$

Considering secants to  $\mathcal{O}$  through  $P$ :

- $(1, 0, 0)' \cap (0, 1, 0)' = (0, 0, 1) = R \in \delta$ ;
- for  $y \in K^*$ ,

$$(1, y, y^{-1})' \cap (1, y + 1, (y + 1)^{-1})' = (1, 1, y^2 + y) \notin \delta.$$



- Since  $P'_0, Q'_0$  meet  $\ell_\infty$  in  $\delta$  they are Baer subplanes of  $\mathcal{H}$  with  $\delta$  removed, so Equation (2) shows that  $\mathcal{H}$  is not tame.
- That the word  $v^{m_1} - v^{m_2}$ , where  $m_1 \cap m_2 = R \in \delta$  is in  $E$  for a specific pair of lines through a specific point in  $\delta$  shows that for every point in  $\delta$  and any pair of lines of  $\Pi$  meeting in that point such a vector will be in  $E$ , since all homologies and elations are present, and the stabilizer of  $\delta$  is transitive on  $\delta$ .
- Thus  $E$  can be completed to  $C_2(\Pi)$  by adding one line through each point of  $\delta$ , so

$$\dim(E) = \dim(C_2(\Pi)) - (q + 1).$$

- $\dim(C_2(\mathcal{H})) > \dim(E) + (q + 1) = \dim(C_2(\Pi))$  since:  
if  $\pi_1, \pi_2 \in \mathcal{L}_n$ , i.e. lines in  $\mathcal{H}$  but Baer subplanes in  $\Pi$ , and  
 $\pi_1 \cap \pi_2 = X_0 \in \ell_\infty^h$ ,  
then  $v^{\pi_1} + v^{\pi_2} \notin E$  since  $\Pi$  is tame, so  $v^{\pi_2} \notin \langle v^{\pi_1}, E \rangle$ .
- The last step can be extended to show

$$\dim(C_2(\mathcal{H})) \geq \dim(C_2(\Pi)) + 2.$$

## Suggested method for $q$ odd

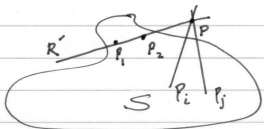
- Let  $w = \sum_{i=1}^m a_i v^{P_i} \in C^\perp$  of weight  $m > 0$ .
- $S = \text{Supp}(w) = \{P_i \mid 1 \leq i \leq m\}$ .
- Suppose  $S$  has a 2-secant,  $P_1 P_2$ , and let  $R = (P_1 P_2)' = P_1' \cap P_2'$ .
- Need to find  $P \in P_1 P_2$ ,  $P \neq P_1, P_2$  such that there is a Baer segment  $\delta$  with  $R \in \delta \subset P'$  with the condition that  $(P P_i)' \notin \delta$  for  $i \geq 3$ , i.e.  $P' \cap P_i' \in P' \setminus \delta$  for  $i \geq 3$ .
- Then

$$a_1(v^{P_2'} - v^{P_1'}) = \sum_{i=3}^m a_i v^{P_i'}.$$

- This is a word in  $\text{Hull}(\mathcal{H})$  that is not the difference of two lines of  $\mathcal{H}$ .

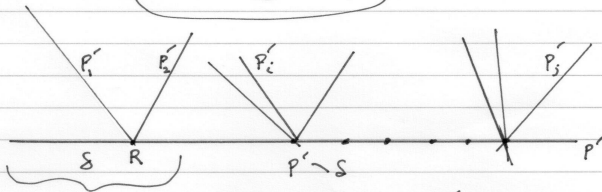


$$W = \sum_{i=1}^m a_i v^{P_i} \in \mathbb{C}^L \quad S = \{P_1, \dots, P_m\}$$



$$a_1 = -a_2$$

$$R' = P_1, P_2 \Rightarrow P$$



$$a_1 (v^{P_2'} - v^{P_1'}) = \sum_{i=3}^m a_i v^{P_i'}$$

We found a suitable word  $w \in C^\perp$  where  $C = C_3(PG_2(\mathbb{F}_9))$ :

$$w = v^{\pi_1} - v^{\pi_2}$$

where  $\pi_1, \pi_2$  are Baer subplanes (i.e.  $PG_2(\mathbb{F}_3)$ ) of  $PG_2(\mathbb{F}_9)$  that meet in a line (Baer segment) and one point off the line.

This word has suitable 2-secants and gives a word of weight 18 in the hull of the Hall plane  $\mathcal{H}$  of order 9 that is not the difference of two lines (of  $\mathcal{H}$ ).

This also shows that  $\dim(C_3(\mathcal{H})) \geq 38$ . (Actual, by computation, is 41).

# Dimension of code of Hall plane of prime square order

The following observation came out of computational results up to  $p = 11$  using the idea of adding incidence vectors of lines to the code  $E$ , and finding the smallest number that must be added at each stage.

## Observation

For  $p$  prime,  $\mathcal{H}_{p^2}$  the projective Hall plane of order  $p^2$ ,  $C = C_p(\mathcal{H}_{p^2})$ ,

$$\dim(C) = \binom{p+1}{2}^2 - p + \sum_{r=2}^p \binom{r}{2} + \binom{p}{2} + 1.$$

(41 for  $p = 3$ .)

From [AK92, Theorem 6.3.3, Corollary 6.4.3]:

## Result

- If  $\pi$  is an affine plane of order  $n$  and  $p|n$ , then **the minimum weight of  $C_p(\pi)$  is  $n$ ; the minimum-weight vectors are constant.**
- If  $\Pi$  is a projective plane of order  $n$  and  $p|n$ , such that *the minimum weight of  $\text{Hull}_p(\Pi) = 2n$* , then **every affine part  $\pi$  of  $\Pi$  has the property that  $C_p(\pi)$  has, as minimum-weight vectors, only the scalar multiples of the incidence vectors of the lines of  $\pi$ .**

# Planes of order 9 and 16

In[GdRK08] the hulls of all the planes of order 9 and all the known planes of order 16 were computed with **Magma** [BCP97, CSW06]. This showed:

- All the non-desarguesian planes of these orders are not tame.
- All the non-translation planes of order 16 have hull of minimum weight 24.
- Most of these words of weight 24 yield words of weight 16 in the binary code of some affine plane of order 16 that are not the incidence vectors of affine lines.

A set of points of a plane has **type  $(n_1, n_2, \dots, n_k)$**  if

- any line meets it in  $n_i$  points for some  $i$ ;
- for each  $i$  there is at least one line that meets it in  $n_i$  points.

# The non-desarguesian planes of order 16

Note:  $\dim(C_2(PG_2(\mathbb{F}_{16}))) = 82$ .

Plane		Dim code	Weight		Type	
SEMI2*		98	32		(0, 2, 4, 8)	
SEMI4*		98	32		(0, 2, 4, 8)	
HALL*	DHALL	98	32	24	(0, 2, 4, 8)	(0, 2, 8)
JOWK*	DJOWK	100	32	24	(0, 2, 4, 8)	(0, 2, 8)
DEMP*	DDEMP	102	32	24	(0, 2, 4, 8)	(0, 2, 8)
LMRH*	DLMRH	106	32	24	(0, 2, 4, 8)	(0, 2, 8)
DSFP*	DDSFP	106	32	24	(0, 2, 4, 8)	(0, 2, 8)
MATH	DMATH	109	24	24	(0, 2, 8)	(0, 2, 8)
BBH1		110	24		(0, 2, 8)	
BBS4	DBBS4	114	24	24	(0, 2, 4)	(0, 2, 8)
JOHN	DJOHN	114	24	24	(0, 2, 8)	(0, 2, 8)
BBH2	DBBH2	114	24	24	(0, 2, 8)	(0, 2, 8)

Table: Minimum words in the hull of non-desarguesian planes of order 16

# Words of weight 16 in affine planes

- All the non-translation projective planes of order 16, apart, possibly, from BBS4, have weight-24 vectors of the type  $(0, 2, 8)$  in the hull.
- If  $w \in \text{Hull}(\Pi)$ ,  $S = \text{Supp}(w)$ ,  $|S| = 24$ ,  $S$  of type  $(0, 2, 8)$  then the three 8-secants meet in a point not in  $S$ , the 8-nucleus of the set.
- If  $\ell$  is an 8-secant to  $S$ , the affine plane  $\pi = \Pi^\ell$  formed by taking  $\ell$  as the line at infinity for  $\Pi$ , has  $v^{S \setminus \ell} \in C_2(\pi)$ , of weight 16,  
i.e. with support two sets of eight points on each of two parallel lines.



# Conclusion







Computations for orders 25, 27, 32 uncovered more planes that are not tame. E.g., all the non-desarguesian translation planes of order 32 are not tame: each has words of weight 64 in the hull whose support set has type  $(0, 2, 4, 16)$ .

We do not have an example yet of a non-desarguesian affine plane of odd order  $n$  divisible by a prime  $p$  whose  $p$ -ary code has vectors of weight  $n$  that are not the incidence vectors of lines.

(All the planes of order 9 have hulls of minimum weight 18, so we need  $n \geq 25$  for an example.)

It might be that a tame plane must be desarguesian as was asked in [AK92, page 238].

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