

# Finite Groups, Designs and Codes -Method 1

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Lecture 5 CIMPA at AIMS: 23 July 2015

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# Abstract

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- We have developed two methods for constructing codes and designs from finite groups (mostly simple finite groups).
- In this talk we first discuss background material and results required from finite groups, permutation groups and representation theory. Then we aim to describe our **first method** of constructing codes and designs from finite groups. This is a survey of the collaborative work by the author with J D Key (Clemson, USA) and B Rorigues (UKZN).

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Error-correcting codes that have large automorphism groups can be useful in applications as the group can help in determining the code's properties, and can be useful in decoding algorithms: see Huffman [15] for a discussion of possibilities, including the question of the use of permutation decoding by searching for PD-sets.

We will discuss two methods for constructing codes and designs for finite groups (mostly simple finite groups).

- In the **first method** we discuss construction of symmetric 1-designs and binary codes obtained from the primitive permutation representations, that is from the action on the maximal subgroups, of a finite group  $G$ .
- This method has been applied to several sporadic simple groups, for example in [18], [22], [23], [27], [28], [29] and [30].

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The **second method** introduces a technique from which a large number of non-symmetric 1-designs could be constructed.

- Let  $G$  be a finite group,  $M$  be a maximal subgroup of  $G$  and  $C_g = [g] = nX$  be the conjugacy class of  $G$  containing  $g$ .
- We construct  $1 - (v, k, \lambda)$  designs  $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ , where  $\mathcal{P} = nX$  and  $\mathcal{B} = \{(M \cap nX)^y \mid y \in G\}$ . The parameters  $v, k, \lambda$  and further properties of  $\mathcal{D}$  are determined.
- We also study codes associated with these designs. In Subsections 5.1, 5.2 and 5.3 we apply the **second method** to the groups  $A_7$ ,  $PSL_2(q)$  and  $J_1$  respectively.

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Our notation will be standard. For finite simple groups and their maximal subgroups we follow the ATLAS notation.

- An **incidence structure**  $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ , with point set  $\mathcal{P}$ , block set  $\mathcal{B}$  and incidence  $\mathcal{I}$  is a  $t$ - $(v, k, \lambda)$  design, if  $|\mathcal{P}| = v$ , every block  $B \in \mathcal{B}$  is incident with precisely  $k$  points, and every  $t$  distinct points are together incident with precisely  $\lambda$  blocks.
- A  $t$ - $(v, k, 1)$  design is called a **Steiner System**. A  $2$ - $(v, 3, 1)$  Steiner system is called a **Steiner Triple System**.

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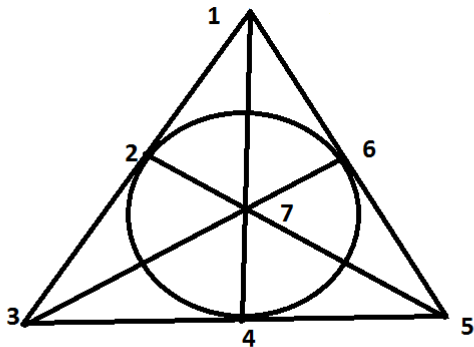
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## Example

The **Fano plane** is a projective plane of order 2, which is a  $2 - (7, 3, 1)$  design (a Steiner triple system on 7 points). Let  $\mathcal{P} = \{1, 2, 3, 4, 5, 6, 7\}$ ,  $\mathcal{B} = \{B_1, B_2, B_3, B_4, B_5, B_6, B_7\}$ , where  $B_1 = \{1, 2, 3\}$ ,  $B_2 = \{1, 5, 6\}$ ,  $B_3 = \{1, 4, 7\}$ ,  $B_4 = \{2, 4, 6\}$ ,  $B_5 = \{2, 5, 7\}$ ,  $B_6 = \{3, 6, 7\}$  and  $B_7 = \{3, 4, 5\}$ .

We can see that the Fano plane is a symmetric 2-design. Also note that it is a  $1 - (7, 3, 3)$  design.

Figure: Fano Plane



- The **complement** of  $\mathcal{D}$  is the structure  $\tilde{\mathcal{D}} = (\mathcal{P}, \mathcal{B}, \tilde{\mathcal{I}})$ , where  $\tilde{\mathcal{I}} = \mathcal{P} \times \mathcal{B} - \mathcal{I}$ . The **dual** structure of  $\mathcal{D}$  is  $\mathcal{D}^t = (\mathcal{B}, \mathcal{P}, \mathcal{I}^t)$ , where  $(B, P) \in \mathcal{I}^t$  if and only if  $(P, B) \in \mathcal{I}$ . Thus the transpose of an **incidence matrix** for  $\mathcal{D}$  is an incidence matrix for  $\mathcal{D}^t$ .
- We will say that the design is symmetric if it has the same number of points and blocks, and self dual if it is isomorphic to its dual.



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- A  $t$ - $(v, k, \lambda)$  design is called **self-orthogonal** if the block intersection numbers have the same parity as the block size.
- The code  $C_F$  of the design  $\mathcal{D}$  over the finite field  $F$  is the space spanned by the incidence vectors of the blocks over  $F$ . We take  $F$  to be a prime field  $F_p$ , in which case we write also  $C_p$  for  $C_F$ , and refer to the dimension of  $C_p$  as the  $p$ -rank of  $\mathcal{D}$ .

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## Example (Incidence Matrix of Fano Plane)

If we let  $M$  be the incidence matrix of the Fano plane, then we have that

$$M = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

It can be shown that  $\text{rank}_F(M) = 7$ , where  $F$  is a field of characteristic  $p$  with  $p \notin \{2, 3\}$ .

$\text{rank}_F(M) = 4$ , where  $\text{char}(F) = 2$  and  $\text{rank}_F(M) = 6$ , where  $\text{char}(F) = 3$ .

- If  $Q$  is any subset of  $\mathcal{P}$ , then we will denote the incidence vector of  $Q$  by  $v^Q$ . Thus  $C_F = \langle v^B \mid B \in \mathcal{B} \rangle$ , and is a subspace of  $F^{\mathcal{P}}$ , the full vector space of functions from  $\mathcal{P}$  to  $F$ .
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- If a linear code over the finite field  $F$  of order  $q$  is of length  $n$ , dimension  $k$ , and minimum weight  $d$ , then we write  $[n, k, d]_q$  to represent this information.
- If  $c$  is a codeword then the **support** of  $c$ ,  $s(c)$ , is the set of non-zero coordinate positions of  $c$ .
- A **constant word** in the code is a codeword all of whose coordinate entries are either 0 or 1. The all-one vector will be denoted by  $\mathbf{j}$ , and is the constant vector of weight the length of the code.
- Two linear codes of the same length and over the same field are **equivalent** if each can be obtained from the other by permuting the coordinate positions and multiplying each coordinate position by a non-zero field element. They are **isomorphic** if they can be obtained from one another by permuting the coordinate positions.

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- An **automorphism** of a code is any permutation of the coordinate positions that maps codewords to codewords. An automorphism thus preserves each weight class of  $C$ . A binary code with all weights divisible by 4 is said to be a **doubly-even** binary code.

Terminology for graphs is standard:

- our graphs are undirected
- the **valency** of a vertex is the number of edges containing the vertex
- A graph is **regular** if all the vertices have the same valence
- a regular graph is **strongly regular** of type  $(n, k, \lambda, \mu)$  if it has  $n$  vertices, valence  $k$ , and if any two adjacent vertices are together adjacent to  $\lambda$  vertices, while any two non-adjacent vertices are together adjacent to  $\mu$  vertices.

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- The groups  $G.H$ ,  $G : H$ , and  $G \ltimes H$  denote a **general extension**, a **split extension (semi-direct product)** and a **non-split extension** respectively.
- For a prime  $p$ ,  $p^n$  denotes the elementary abelian group of order  $p^n$ , that is  $\mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ ,  $n$  copies.
- If  $G$  is a permutation group on  $\Omega = \{1, 2, \dots, n\}$  and  $M$  is a group, then the wreath product  $M \wr G$ , is the split extension  $M^n : G$ , where

$$M^n = M \times M \times \cdots \times M = \{(m_1, m_2, \dots, m_n) \mid m_i \in M\},$$

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## CFSG Theorem

The **classification of finite simple groups** was completed in 1981. It has a history of nearly 150 years and its proof occupies 15000 journal pages. The classification theorem (CFSG) is precisely:

*Every finite simple group is isomorphic to one of the following groups*

- *a group of prime order,*
- *an alternating group  $A_n$  for  $n \geq 5$ ,*
- *one of the finite groups of Lie type (classical or exceptional),*
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Suppose that  $G$  is a finite group acting on a finite set  $\Omega$ . Let  $\alpha \in \Omega$ , then

Then  $G_\alpha \leq G$  and  $[G : G_\alpha] = |\Delta|$ , where  $\Delta$  is the orbit containing  $\alpha$ .

The action of  $G$  on  $\Omega$  gives a permutation representation  $\pi$  with corresponding permutation character  $\chi_\pi$  denoted by  $\chi(G|\Omega)$ .

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## Lemma

- (i) *The action of  $G$  on  $\Omega$  is isomorphic to the action of  $G$  on the  $G/G_\alpha$ , that is on the set of all left cosets of  $G_\alpha$  in  $G$ . Hence  $\chi(G|\Omega) = \chi(G|G_\alpha)$ .*
- (ii)  $\chi(G|\Omega) = (I_{G_\alpha})^G$ , *the trivial character of  $G_\alpha$  induced to  $G$ .*
- (iii) *For all  $g \in G$ , we have  $\chi(G|\Omega)(g) =$  number of points in  $\Omega$  fixed by  $g$ .*

**Proof:** For example see Isaacs [11] or Ali [1]. ■

In fact for any subgroup  $H \leq G$  we have

$$\chi(G|H)(g) = \sum_{i=1}^k \frac{|C_G(g)|}{|C_H(h_i)|},$$

$h_i$ 's are rep. of the conj. classes of  $H$  that fuse to  $[g] = C_g$  in  $G$ . 


## Lemma

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Now the results follows from Lemma 4.1 part (i).

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## Corrolary

*If  $G$  is a finite simple group and  $M$  is a maximal subgroup of  $G$ , then number  $\lambda$  of conjugates of  $M$  in  $G$  containing  $g$  is given by*

$$\chi(G|M)(g) = \sum_{i=1}^k \frac{|C_G(g)|}{|C_M(x_i)|},$$

*where  $x_1, x_2, \dots, x_k$  are representatives of the conjugacy classes of  $M$  that fuse to the class  $[g] = C_g$  in  $G$ .*

**Proof:** It follows from Lemma 4.2 and the fact that  $N_G(M) = M$ . It is also a direct application of Remark 1, since

$$\chi(G|\Omega)(g) = |\{M^x | g \in (N_G(M))^x\}| = |\{M^x | g \in M^x\}|. \blacksquare$$

Let  $B$  be a subset of  $\Omega$ . If  $B^g = B$  or  $B^g \cap B = \emptyset$  for all  $g \in G$ , we say  $B$  is a **block** for  $G$ . Clearly  $\emptyset, \Omega$  and  $\{\alpha\}$  for all  $\alpha \in \Omega$  are blocks, called **trivial blocks**. Any other block is called **non-trivial**. If  $G$  is transitive on  $\Omega$  such that  $G$  has no non-trivial block on  $\Omega$ , then we say  $G$  is **primitive**. Otherwise we say  $G$  is **imprimitive**.

- Classification of Finite Simple Groups (CFSG) implies that no 6-transitive finite groups exist other than  $S_n$  ( $n \geq 6$ ) and  $A_n$  ( $n \geq 8$ ), and that the Mathieu groups are the only faithful permutation groups other than  $S_n$  and  $A_n$  providing examples for 4- and 5-transitive groups.
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The following is a well-known theorem that gives a characterisation of primitive permutation groups.

Since by Lemma 4.1 the permutation action of a group  $G$  on a set  $\Omega$  is equivalent to the action of  $G$  on the set of the left cosets  $G/G_\alpha$ , determination of the primitive actions of  $G$  reduces to the classification of its maximal subgroups.

### Theorem

*Let  $G$  be transitive permutation group on a set  $\Omega$ . Then  $G$  is primitive if and only if  $G_\alpha$  is a maximal subgroup of  $G$  for every  $\alpha \in \Omega$ .*

**Proof:** See Rotman [33]. ■

If  $G$  is transitive on  $\Omega$  and  $G_\alpha$  has  $r$  orbits on  $\Omega$ , then we say that  $G$  is a **rank- $r$**  permutation group.

- Currently the primitive permutation groups of degree  $n$  with  $n < 1000$  and primitive solvable permutation groups of degree less than 6561 have been classified (see [14]). Most of the computational procedures have been implemented in MAGMA [4] and GAP [12].

# Construction of 1-Designs and Codes from Maximal Subgroups

In this section we consider primitive representations of a finite group  $G$ . Let  $G$  be a finite primitive permutation group acting on the set  $\Omega$  of size  $n$ . We can consider the action of  $G$  on  $\Omega \times \Omega$  given by  $(\alpha, \beta)^g = (\alpha^g, \beta^g)$  for all  $\alpha, \beta \in \Omega$  and all  $g \in G$ . An orbit of  $G$  on  $\Omega \times \Omega$  is called an **orbital**. If  $\bar{\Delta}$  is an orbital, then  $\bar{\Delta}^* = \{(\alpha, \beta) : (\beta, \alpha) \in \bar{\Delta}\}$  is also an orbital of  $G$  on  $\Omega \times \Omega$ , which is called the **paired orbital** of  $\bar{\Delta}$ . We say that  $\bar{\Delta}$  is **self-paired** if  $\bar{\Delta} = \bar{\Delta}^*$ .

For  $\alpha \in \Omega$ , let  $\Delta \neq \{\alpha\}$  be an orbit of the stabilizer  $M = G_\alpha$  of  $\alpha$ . Then  $\bar{\Delta}$  given by  $\bar{\Delta} = \{(\alpha, \delta)^g : \delta \in \Delta, g \in G\}$  is an orbital. We say that  $\Delta$  is self-paired if and only if  $\bar{\Delta}$  is a self paired orbital. The primitivity of  $G$  on  $\Omega$  implies that  $M$  is maximal in  $G$ .

Our construction for the symmetric 1-designs is based on the following results, mainly Theorem 5.1 below, which is the Proposition 1 of [18] with its corrected version in [19]:

### Theorem

*Let  $G$  be a finite primitive permutation group acting on the set  $\Omega$  of size  $n$ . Let  $\alpha \in \Omega$ , and let  $\Delta \neq \{\alpha\}$  be an orbit of the stabilizer  $G_\alpha$  of  $\alpha$ . If  $\mathcal{B} = \{\Delta^g : g \in G\}$  and, given  $\delta \in \Delta$ ,  $\mathcal{E} = \{\{\alpha, \delta\}^g : g \in G\}$ , then  $\mathcal{D} = (\Omega, \mathcal{B})$  forms a 1- $(n, |\Delta|, |\Delta|)$  design with  $n$  blocks. Further, if  $\Delta$  is a self-paired orbit of  $G_\alpha$ , then  $\Gamma = (\Omega, \mathcal{E})$  is a regular connected graph of valency  $|\Delta|$ ,  $\mathcal{D}$  is self-dual, and  $G$  acts as an automorphism group on each of these structures, primitive on vertices of the graph, and on points and blocks of the design.*

**Proof:** We have  $|G| = |\Delta^G| |G_\Delta|$ , and clearly  $G_\Delta \supseteq G_\alpha$ . Since  $G$  is primitive on  $\Omega$ ,  $G_\alpha$  is maximal in  $G$ , and thus  $G_\Delta = G_\alpha$ , and  $|\Delta^G| = |\mathcal{B}| = n$ . This proves that we have a  $1$ - $(n, |\Delta|, |\Delta|)$  design. Since  $\Delta$  is self-paired,  $\Gamma$  is a graph rather than only a digraph. In  $\Gamma$  we notice that the vertices adjacent to  $\alpha$  are the vertices in  $\Delta$ . Now as we orbit these pairs under  $G$ , we get the  $nk$  ordered pairs, and thus  $nk/2$  edges, where  $k = |\Delta|$ . Since the graph has  $G$  acting, it is clearly regular, and thus the valency is  $k$  as required, i.e. the only vertices adjacent to  $\alpha$  are those in the orbit  $\Delta$ . The graph must be connected, as a maximal connected component will form a block of imprimitivity, contradicting the group's primitive action. Now notice that an adjacency matrix for the graph is simply an incidence matrix for the  $1$ -design, so that the  $1$ -design is necessarily self-dual. This proves all our assertions. ■

Note that if we form any union of orbits of  $G_\alpha$ , including the orbit  $\{\alpha\}$ , and orbit this under the full group, we will still get a self-dual symmetric 1-design with the group operating. Thus the orbits of the stabilizer can be regarded as “building blocks”. Since the complementary design (i.e. taking the complements of the blocks to be the new blocks) will have exactly the same properties, we will assume that our block size is at most  $v/2$ . In fact this will give us all possible designs on which the group acts primitively on points and blocks:

### Lemma

*If the group  $G$  acts primitively on the points and the blocks of a symmetric 1-design  $\mathcal{D}$ , then the design can be obtained by orbiting a union of orbits of a point-stabilizer, as described in Theorem 5.1.*

**Proof:** Suppose that  $G$  acts primitively on points and blocks of the  $1-(v, k, k)$  design  $\mathcal{D}$ . Let  $\mathcal{B}$  be the block set of  $\mathcal{D}$ ; then if  $B$  is any block of  $\mathcal{D}$ ,  $\mathcal{B} = B^G$ . Thus  $|G| = |\mathcal{B}||G_B|$ , and since  $G$  is primitive,  $G_B$  is maximal and thus  $G_B = G_\alpha$  for some point. Thus  $G_\alpha$  fixes  $B$ , so this must be a union of orbits of  $G_\alpha$ . ■

## Lemma

*If  $G$  is a primitive simple group acting on  $\Omega$ , then for any  $\alpha \in \Omega$ , the point stabilizer  $G_\alpha$  has only one orbit of length 1.*

**Proof:** Suppose that  $G_\alpha$  fixes also  $\beta$ . Then  $G_\alpha = G_\beta$ . Since  $G$  is transitive, there exists  $g \in G$  such that  $\alpha^g = \beta$ . Then  $(G_\alpha)^g = G_{\alpha^g} = G_\beta = G_\alpha$ , and thus  $g \in N_G(G_\alpha) = N$ . Since  $G_\alpha$  is maximal in  $G$ , we have  $N = G$  or  $N = G_\alpha$ . But  $G$  is simple, so we must have  $N = G_\alpha$ , so that  $g \in G_\alpha$  and so  $\beta = \alpha$ . ■



- We have considered various finite simple groups, for example  $J_1$ ;  $J_2$ ;  $M^cL$ ;  $PSp_{2m}(q)$ , where  $q$  is a power of an odd prime, and  $m \geq 2$ ;  $Co_2$ ;  $HS$  and  $Ru$ .
- For each group, using Magma [4], we construct designs and graphs that have the group acting primitively on points as automorphism group, and, for a selection of small primes, codes over that prime field derived from the designs or graphs that also have the group acting as automorphism group. For each code, the code automorphism group at least contains the associated group  $G$ .
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- It is well known, and easy to see, that if the group is rank-3, then the graph formed as described in Theorem 5.1 will be strongly regular. In case the group is not of rank 3, this might still happen, and we examined this question also for some of the groups we studied.
- Clearly  $G \leq \text{Aut}(D) \leq \text{Aut}(C)$ . Note that we could in some cases look for the full group of the hull, and from that deduce the group of the code, since  $\text{Aut}(C) = \text{Aut}(C^\perp) \subseteq \text{Aut}(C \cap C^\perp)$ .
- A sample of our results for example for  $J_1$  and  $J_2$  is given below. We looked at some of the codes that were computationally feasible to find out if the groups  $J_1$  and  $\text{Aut}(J_2) = J_2 : 2 = \bar{J}_2$  formed the full automorphism group in any of the cases when the code was not the full vector space. We first mention the following lemma

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## Lemma

*Let  $C$  be the linear code of length  $n$  of an incidence structure  $\mathcal{I}$  over a field  $F$ . Then the automorphism group of  $C$  is the full symmetric group if and only if  $C = F^n$  or  $C = F_J^\perp$ .*

**Proof:** Suppose  $\text{Aut}(C)$  is  $S_n$ . Then  $C$  is spanned by the incidence vectors of the blocks of  $\mathcal{I}$ ; let  $B$  be such a block and suppose it has  $k$  points, and so it gives a vector of weight  $k$  in  $C$ . Clearly  $C$  contains the incidence vector of any set of  $k$  points, and thus, by taking the difference of two such vectors that differ in just two places, we see that  $C$  contains all the vectors of weight 2 having as non-zero entries 1 and  $-1$ . Thus  $C = F_J^\perp$  or  $F^n$ . The converse is clear. ■

Here we give a brief discussion on the application of Method 1 to the sporadic simple groups  $J_1$ ,  $J_2$  and  $Co_2$ . For full details the readers are referred to [18], [19], [20] and [28].

### Computations for $J_1$ and $J_2$

- The first Janko sporadic simple group  $J_1$  has order  $175560 = 2^3 \times 3 \times 5 \times 7 \times 11 \times 19$  and it has seven distinct primitive representations, of degree 266, 1045, 1463, 1540, 1596, 2926, and 4180, respectively (see Table 1 and [5, 9]).
- For each of the seven primitive representations, using Magma, we constructed the permutation group and formed the orbits of the stabilizer of a point. For each of the non-trivial orbits, we formed the symmetric 1-design as described in Theorem 5.1.



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- We also found the automorphism group of each design, which will be the same as the automorphism group of the regular graph. Where computationally possible we also found the automorphism group of the code.
- Conclusions from our results are summarized below. In brief, we found that there are 245 designs formed in this manner from single orbits and that none of them is isomorphic to any other of the designs in this set. In every case the full automorphism group of the design or graph is  $J_1$ .

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Table 1: Maximal subgroups of  $J_1$

No.	Order	Index	Structure
Max[1]	660	266	$PSL(2, 11)$
Max[2]	168	1045	$2^3:7:3$
Max[3]	120	1463	$2 \times A_5$
Max[4]	114	1540	19:6
Max[5]	110	1596	11:10
Max[6]	60	2926	$D_6 \times D_{10}$
Max[7]	42	4180	7:6

In Table 2, 1st column gives the degree, 2nd the number of orbits, and the remaining columns give the length of the orbits of length greater than 1 (with the number of that length in case there is more than one of that length).

Table 2: Orbits of a point-stabilizer of  $J_1$

Degree	#	length				
266	5	132	110	12	11	
1045	11	168(5)	56(3)	28	8	
1463	22	120(7)	60(9)	20(2)	15(2)	12
1540	21	114(9)	57(6)	38(4)	19	
1596	19	110(13)	55(2)	22(2)	11	
2926	67	60(34)	30(27)	15(5)		
4180	107	42(95)	21(6)	14(4)	7	

In summary we have the following result:

## Proposition

*If  $G$  is the first Janko group  $J_1$ , there are precisely 245 non-isomorphic self-dual 1-designs obtained by taking all the images under  $G$  of the non-trivial orbits of the point stabilizer in any of  $G$ 's primitive representations, and on which  $G$  acts primitively on points and blocks. In each case the full automorphism group is  $J_1$ . Every primitive action on symmetric 1-designs can be obtained by taking the union of such orbits and orbiting under  $G$ .*

We tested the graphs for strong regularity in the cases of the smaller degree, and did not find any that were strongly regular. We also found the designs and their codes for some of the unions of orbits in some cases.

- The second Janko sporadic simple group  $J_2$  has order  $604800 = 2^7 \times 3^3 \times 5^2 \times 7$ , and it has nine primitive permutation representations (see Table 3), but we did not compute with the largest degree.
- Our results for  $J_2$  are different from those for  $J_1$ , due to the existence of an outer automorphism. The main difference is that usually the full automorphism group is  $\bar{J}_2 = J_2 : 2$ , and that in the cases where it was only  $J_2$ , there would be another orbit of that length that would give an isomorphic design, and which, if the two orbits were joined, would give a design of double the block size and automorphism group  $\bar{J}_2$ . A similar conclusion held if some union of orbits was taken as a base block.



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Table 3: Maximal subgroups of  $J_2$

No.	Order	Index	Structure
Max[1]	6048	100	$PSU(3, 3)$
Max[2]	2160	280	$3 \cdot PGL(2, 9)$
Max[3]	1920	315	$2^{1+4} : A_5$
Max[4]	1152	525	$2^{2+4} : (3 \times S_3)$
Max[5]	720	840	$A_4 \times A_5$
Max[6]	600	1008	$A_5 \times D_{10}$
Max[7]	336	1800	$PSL(2, 7) : 2$
Max[8]	300	2016	$5^2 : D_{12}$
Max[9]	60	10080	$A_5$

Table 4: Orbits of a point-stabilizer of  $J_2$  (of degree  $\leq 2016$ )

Degree	#	length						
100	3	63	36					
280	4	135	108	36				
315	6	160	80	32(2)	10			
525	6	192(2)	96	32	12			
840	7	360	240	180	24	20	15	
1008	11	300	150(2)	100(2)	60(2)	50	25	12
1800	18	336	168(6)	84(3)	42(3)	28	21	14(2)
2016	18	300(2)	150(6)	75(5)	50(2)	25	15	

From these eight primitive representations, we obtained in all 51 non-isomorphic symmetric designs on which  $J_2$  acts primitively.

We also found three strongly regular graphs (all of which are known: see Brouwer [6]): that of degree 100 from the rank-3 action, of course, and two more of degree 280 from the orbits of length 135 and 36, giving strongly regular graphs with parameters  $(280,135,70,60)$  and  $(280,36,8,4)$  respectively. The full automorphism group is  $\bar{J}_2$  in each case.

In each of the following we consider the primitive action of  $J_2$  on a design formed as described in Method 1 from an orbit or a union of orbits, and the codes are the codes of the associated 1-design.

- For  $J_2$  of degree 100,  $\bar{J}_2$  is the full automorphism group of the design with parameters  $1-(100, 36, 36)$ , and it is the automorphism group of the self-orthogonal doubly-even  $[100, 36, 16]_2$  binary code of this design.
- For  $J_2$  of degree 280,  $\bar{J}_2$  is the full automorphism group of the design with parameters  $1-(280, 108, 108)$ , and it is the automorphism group of the self-orthogonal doubly-even  $[280, 14, 108]_2$  binary code of this design. The weight distribution of this code is

$\langle 0, 1 \rangle, \langle 108, 280 \rangle, \langle 128, 1575 \rangle, \langle 136, 2520 \rangle, \langle 140, 7632 \rangle, \langle 144, 2520 \rangle,$   
 $\langle 152, 1575 \rangle, \langle 172, 280 \rangle, \langle 280, 1 \rangle$

Thus the words of minimum weight (i.e. 108) are the incidence vectors of the design.

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- For  $J_2$  of degree 280,  $\bar{J}_2$  is the full automorphism group of the design with parameters  $1-(280, 108, 108)$ , and it is the automorphism group of the self-orthogonal doubly-even  $[280, 14, 108]_2$  binary code of this design. The weight distribution of this code is

$\langle 0, 1 \rangle, \langle 108, 280 \rangle, \langle 128, 1575 \rangle, \langle 136, 2520 \rangle, \langle 140, 7632 \rangle, \langle 144, 2520 \rangle,$   
 $\langle 152, 1575 \rangle, \langle 172, 280 \rangle, \langle 280, 1 \rangle$

Thus the words of minimum weight (i.e. 108) are the incidence vectors of the design.

- For  $J_2$  of degree 315,  $\bar{J}_2$  is the full automorphism group of the design with parameters  $1-(315, 64, 64)$  (by taking the union of the two orbits of length 32), and it is the automorphism group of the self orthogonal doubly-even  $[315, 28, 64]_2$  binary code of this design. The weight distribution of the code is as follows:

$\langle 0, 1 \rangle, \langle 64, 315 \rangle, \langle 96, 6300 \rangle, \langle 104, 25200 \rangle, \langle 112, 53280 \rangle, \langle 120, 242760 \rangle,$   
 $\langle 124, 201600 \rangle, \langle 128, 875700 \rangle, \langle 132, 1733760 \rangle, \langle 136, 4158000 \rangle, \langle 140, 5973120 \rangle,$   
 $\langle 144, 12626880 \rangle, \langle 148, 24232320 \rangle, \langle 152, 35151480 \rangle, \langle 156, 44392320 \rangle,$   
 $\langle 160, 53040582 \rangle, \langle 164, 41731200 \rangle, \langle 168, 28065120 \rangle, \langle 172, 13023360 \rangle,$   
 $\langle 176, 2129400 \rangle, \langle 180, 685440 \rangle, \langle 184, 75600 \rangle, \langle 192, 10710 \rangle, \langle 200, 1008 \rangle$

Thus the words of minimum weight (i.e. 64) are the incidence vectors of the blocks of the design.

- Furthermore, the designs from the two orbits of length 32 in this case, i.e.  $1-(315, 32, 32)$  designs, each have  $J_2$  as their automorphism group. Their binary codes are equal, and are  $[315, 188]_2$  codes, with hull the 28-dimensional code described above. The automorphism group of this 188-dimensional code is again  $\bar{J}_2$ . The minimum weight is at most 32.
- For  $J_2$  of degree 315,  $\bar{J}_2$  is the full automorphism group of the design with parameters  $1-(315, 160, 160)$  and it is the automorphism group of the  $[315, 265]_5$  5-ary code of this design. This code is also the 5-ary code of the design obtained from the orbit of length 10, and from that of the orbit of length 80, so we can deduce that the minimum weight is at most 10. The hull is a  $[315, 15, 155]_5$  code and again with  $\bar{J}_2$  as full automorphism group.



- Furthermore, the designs from the two orbits of length 32 in this case, i.e.  $1-(315, 32, 32)$  designs, each have  $J_2$  as their automorphism group. Their binary codes are equal, and are  $[315, 188]_2$  codes, with hull the 28-dimensional code described above. The automorphism group of this 188-dimensional code is again  $\bar{J}_2$ . The minimum weight is at most 32.
- For  $J_2$  of degree 315,  $\bar{J}_2$  is the full automorphism group of the design with parameters  $1-(315, 160, 160)$  and it is the automorphism group of the  $[315, 265]_5$  5-ary code of this design. This code is also the 5-ary code of the design obtained from the orbit of length 10, and from that of the orbit of length 80, so we can deduce that the minimum weight is at most 10. The hull is a  $[315, 15, 155]_5$  code and again with  $\bar{J}_2$  as full automorphism group.

- For  $J_2$  of degree 315,  $\bar{J}_2$  is the full automorphism group of the design with parameters 1-(315, 80, 80) from the orbit of length 80, and it is the automorphism group of the self-orthogonal doubly-even  $[315, 36, 80]_2$  binary code of this design. The minimum words of this code are precisely the 315 incidence vectors of the blocks of the design.

**Irreducible Modules of  $J_1$  and  $J_2$ :** In [20] we used Method 1 to obtain all irreducible modules of  $J_1$  (as codes) over  $\mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_5$ . Most of irreducible modules of  $J_2$  can be represented in this way as the code, the dual code or the hull of the code of a design, or of codimension 1 in one of these. For  $J_2$ , if no such code was found for a particular irreducible module, then we checked that it could not be so represented for the relevant degrees of the primitive permutation representations up to and including 1008. In summary, we obtained:

## Proposition

*Using the construction described in Method 1 above (see Theorem 5.1 and Lemma 5.2), taking unions of orbits, the following constructions of the irreducible modules of the Janko groups  $J_1$  and  $J_2$  as the code, the dual code or the hull of the code of a design, or of codimension 1 in one of these, over  $\mathbb{F}_p$  where  $p = 2, 3, 5$ , were found to be possible:*

- 1  $J_1$ : all the seven irreducible modules for  $p = 2, 3, 5$ ;
- 2  $J_2$ : all for  $p = 2$  apart from dimensions 12, 128; all for  $p = 3$  apart from dimensions 26, 42, 114, 378; all for  $p = 5$  apart from dimensions 21, 70, 189, 300. For these exclusions, none exist of degree  $\leq 1008$ .

## Proposition

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## Notes

- We do not claim that we have all the constructions of the modular representations as codes; we were seeking mainly existence.
- In the tables, the row labelled “Dim” denotes the dimensions of the distinct irreducible modules, and the row labelled “Deg” denotes the degree of the permutation representation i.e. the length of the code. An entry “–” indicates that none were found for that dimension, and that none of degree  $\leq 1008$  exist.

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# Codes of irreducible modules of $J_1$ for $p = 2, 3, 5$

$p = 2$	Dim	20	76	76
	Deg	1045, 1463, 1540	266, 1045, 1463	1463
	Dim	112	112	360
	Deg	266, 1045	1463	1045

$p = 3$	Dim	76	76	112	133
	Deg	266, 1045, 1596	1596	266, 1045	1045
	Dim	154	360		
	Deg	1045	1045		

$p = 5$	Dim	56	76	76	77	133	360
	Deg	266	1045	1596	266	1596	1045



We constructed three self-orthogonal binary codes of dimension 20 invariant under  $J_1$  of lengths 1045, 1463, and 1540. These are irreducible by [16] or Magma data. The Magma *simgps* library is used. In the following we only discuss one of these:  $J_1$  of Degree 1045 - Code:  $[1045, 20, 456]_2$  Dual Code:  $[1045, 1025, 4]_2$

- Permutation group  $J_1$  acting on a set of cardinality 1045
- Orbit lengths of stabilizer of a point:  $[1, 8, 28, 56, 56, 56, 168, 168, 168, 168, 168]$ ;
- Orbits chosen: 1,3,5,10,11. Defining block is the union of these orbits, length 421
- $1 - (1045, 421, 421)$  Design with 1045 blocks
- $C$  is the code of the design, of dimension 21
- The 20-dimensional code is  $C \cap C^\perp = \text{Hull}(C)$
- $C = \text{Hull}(C) \oplus \langle y \rangle$ , has type  $[1045, 21, 421]$

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- The full space can be completely decomposed into  $J_1$ -modules:  $V = \mathbb{F}_2^{1045} = C_{76} \oplus C_{112} \oplus C_{360} \oplus C_{496} \oplus C_1$ , where all but  $C_{496}$  are irreducible.  $C_{496}$  has composition factors of dimensions:  
 $20, 112, 1, 76, 20, 1, 112, 20, 1, 1, 112, 20$ .  
 Note that  $Soc(V) = Hull(C) \oplus \langle j \rangle \oplus C_{76} \oplus C_{112} \oplus C_{360}$ , with  $dim(Soc(V)) = 569$ .
- Weight Distribution of  $Hull(C)$ :  $\langle 0, 1 \rangle, \langle 456, 3080 \rangle, \langle 488, 29260 \rangle, \langle 496, 87780 \rangle, \langle 504, 87780 \rangle, \langle 512, 36575 \rangle, \langle 520, 299706 \rangle, \langle 528, 234080 \rangle, \langle 536, 175560 \rangle, \langle 544, 58520 \rangle, \langle 552, 14630 \rangle, \langle 560, 19019 \rangle, \langle 608, 1540 \rangle, \langle 624, 1045 \rangle$ .



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 $20, 112, 1, 76, 20, 1, 112, 20, 1, 1, 112, 20$ .  
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- **Weight Distribution of C:**  $\langle 0, 1 \rangle$ ,  $\langle 421, 1405 \rangle$ ,  
 $\langle 437, 1540 \rangle$ ,  $\langle 456, 3080 \rangle$ ,  $\langle 485, 19019 \rangle$ ,  
 $\langle 488, 29260 \rangle$ ,  $\langle 493, 14630 \rangle$ ,  $\langle 496, 87780 \rangle$ ,  
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 $\langle 549, 87780 \rangle$ ,  $\langle 552, 14630 \rangle$ ,  $\langle 557, 29260 \rangle$ ,  
 $\langle 560, 19019 \rangle$ ,  $\langle 589, 3080 \rangle$ ,  $\langle 608, 1540 \rangle$ ,  
 $\langle 624, 1045 \rangle$ ,  $\langle 1045, 1 \rangle$ .

# Codes of irreducible modules of $J_2$ for $p = 2, 3, 5$

$p = 2$	Dim	12	28	36	84	128	160
	Deg	–	315	100	840	–	315

$p = 3$	Dim	26	36	42	63	90	114
	Deg	–	100	–	100	280	–
	Dim	133	225	378			
	Deg	525	1008	–			

$p = 5$	Dim	14	21	41	70	85	90	175
	Deg	315	–	280	–	1008	315	525
	Dim	189	225	300				
	Deg	–	840	–				

We now look at the smallest representations for  $J_2$ . We have not been able to find any of dimension 12, and none can exist for degree  $\leq 1008$ , as we have verified computationally by examining the permutation modules.

We give below four representations of  $J_2$  acting on self-orthogonal binary codes of small degree that are irreducible or indecomposable codes over  $J_2$ .

The full automorphism group of each of these codes is  $\bar{J}_2$ .

Degree 100, dimension 36, code  $[100, 36, 16]_2$  ; dual code:  $[100, 64, 8]_2$

- Permutation group  $J_2$  acting on a set of cardinality 100
- Orbit lengths of stabilizer of a point: 1, 36, 63
- 1-(100, 36, 36) Design with 100 blocks
- Second orbit gave a block of the design
- $C = C_{36}$  is the code of the design of dimension 36,  $Aut(C) = \bar{J}_2$ , and it is irreducible.
- $C_{36}$  has type  $[100, 36, 16]_2$
- Weigh distribution of  $C_{36}$  has been determined
- $C_{64} = C^\perp$  contains  $C_{36}$  and  $\langle \gamma \rangle$ , but it is indecomposable
- $V = \mathbb{F}_2^{100}$  is indecomposable. Also  $Soc(V) = C_{36} \oplus \langle \gamma \rangle$

Degree 100, dimension 36, code  $[100, 36, 16]_2$  ; dual code:  $[100, 64, 8]_2$

- Permutation group  $J_2$  acting on a set of cardinality 100
- Orbit lengths of stabilizer of a point: **1, 36, 63**
- 1-(100, 36, 36) Design with 100 blocks
- Second orbit gave a block of the design
- $C = C_{36}$  is the code of the design of dimension 36,  $Aut(C) = \bar{J}_2$ , and it is irreducible.
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Degree 100, dimension 36, code  $[100, 36, 16]_2$  ; dual code:  $[100, 64, 8]_2$

- Permutation group  $J_2$  acting on a set of cardinality 100
- Orbit lengths of stabilizer of a point: **1, 36, 63**
- **1-(100, 36, 36)** Design with 100 blocks
- Second orbit gave a block of the design
- $C = C_{36}$  is the code of the design of dimension 36,  $Aut(C) = \bar{J}_2$ , and it is irreducible.
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- Permutation group  $J_2$  acting on a set of cardinality 315
- Orbit lengths a point stabilizer:  $[1, 10, 32, 32, 80, 160]$
- Orbits chosen: 3 and 4
- 1-(315, 64, 64) Design with 315 blocks
- $C = C_{28}$  is the code of the design of dimension 28, it is irreducible,  $Aut(C) = \bar{J}_2$ .
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- We give a brief discussion of the Conway group  $Co_2$ . The group  $Co_2$  admits a **23-dimensional indecomposable representation** (say  $M$ ) over  $GF(2)$  obtained from the 24-dimensional Leech lattice by reducing modulo 2 and factoring out a fixed vector.

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- On the other hand, reduction modulo 2 of the 23-dimensional ordinary irreducible representation results in a **decomposable 23-dimensional**  $GF(2)$ -representation (say  $L$ ). We construct this decomposable 23-dimensional  $GF(2)$ -representation as a binary code.
- Furthermore, we show that this code contains a binary code of dimension 22 invariant and irreducible under the action of  $Co_2$ .



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# $S(5, 8, 24)$

## Octads and Dodecads

Let  $\Omega = \{1, 2, 3, \dots, 24\}$ . Consider the Steiner system  $S(5, 8, 24)$  on this set. Each block is called an **Octad** and is denoted by  $8^\circ$ .

- There are 759 octads.
- Any two octads  $O_1$  and  $O_2$  intersect in a set of cardinality 0, 2, 4 or 8
- If  $|O_1 \cap O_2| = 2$ , then  $O_1 \Delta O_2$  is called a **dodecad** and is denoted by  $12^\circ$ .
- There are 2576 dodecads in  $S(5, 8, 24)$ .

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# Leech Lattice

The **Leech lattice**  $\Lambda$  was discovered by John Leech (1926–1992), in three papers written in 1964, 1965 and 1967, in connection with **close packing of spheres** in 24 dimension.  $\Lambda$  consists of  $(x_1, x_2, \dots, x_{24}) \in \mathbb{Z}^{24}$  such that

- (i)  $\sum_{i=1}^{24} x_i \equiv 4m \pmod{8}$
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## Leech Lattice 2

If  $(, )$  denotes the Euclidean bilinear form on  $\mathbb{R}^{24}$ . Then for all  $x, y \in \Lambda$  we have

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# The Conway Group $.0 = Co_0$

The Leech group (Conway group  $.0$  in 1967) is the  $Aut(\Lambda)$ .  
Conway proved that

- (i)  $N = 2^{12}.M_{24}$  is a maximal subgroup of  $.0$
- (ii)  $|.0| = 2^{22}3^95^47^211 \times 13 \times 23$ .
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We define  $\Lambda_n$  by

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$$\lambda_i \in \Lambda_i$$

- Many other sporadic simple groups can be constructed as the stabilizers.

## $.0 = Co_0$ Action on $\Lambda$

We define  $\Lambda_n$  by

$$\Lambda_n = \{x \in \Lambda : \|x\| = 4\sqrt{n}\}.$$

Then  $.0$  acts transitively on  $\Lambda_i, i = 2, 3, 4$ .

- (i)  $|\Lambda_2| = 196560$ ,  $(.0)_{\lambda_2} = .2 = Co_2$  **new simple group**
- (ii)  $|\Lambda_3| = 16737120$ ,  $(.0)_{\lambda_3} = .3 = Co_3$  **new simple group**
- (iii)  $|\Lambda_4| = 398034000$ ,  $(.0)_{\lambda_4} = .4 = 2^{11}.M_{23}$  **not simple**
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## Conway Group $Co_2$

- The group  $Co_2$  admits a 23-dimensional indecomposable representation over  $GF(2)$  obtained from the 24-dimensional Leech lattice by reducing modulo 2 and factoring out a fixed vector. The action of  $Co_2$  on the vectors of this 23-dimensional indecomposable  $GF(2)$ -module (say  $M$ ) produces eight orbits.
- $M$  contains an irreducible  $GF(2)$ -submodule  $N$  of dimension 22.
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Table 5: Action of  $Co_2$  on  $M$  and  $N$

$M$ -Stabilizer	$M$ -Orbit length	$N$ -Stabilizer	$N$ -Orbit length
$Co_2$	1	$Co_2$	1
$U_6(2) : 2$	2300	$U_6(2) : 2$	2300
$M^cL$	47104		
$2^{10}:M_{22}:2$	46575	$2^{10}:M_{22}:2$	46575
$HS:2$	476928	$HS:2$	476928
$U_4(3).D_8$	1619200	$U_4(3).D_8$	1619200
$M_{23}$	4147200		
$2_+^{1+8}:S_8$	2049300	$2_+^{1+8}:S_8$	2049300



## Maximal subgroups of $Co_2$

No.	Max. sub.	Deg.
1	$U_6(2):2$	2300
2	$2^{10}:M_{22}:2$	46575
3	$M^cL$	47104
4	$2_+^{1+8}:S_6(2)$	56925
5	$HS:2$	476928
6	$(2_+^{1+6} \times 2^4) \cdot A_8$	1024650
7	$U_4(3) \cdot D_8$	1619200
8	$2^{4+10}(S_5 \times S_3)$	3586275
9	$M_{23}$	4147200
10	$3_+^{1+4}:2_-^{1+4} \cdot S_5$	45337600
11	$5_+^{1+2}4S_4$	3525451776

# Permutation Representation of Degree 2300

- $Co_2$  acts on the left cosets of  $U_6(2):2$  as a **rank-3 primitive permutation representation** of degree 2300.
- The stabilizer of a point  $\alpha$  in this representation is a maximal subgroup isomorphic to  $U_6(2):2$ , producing three orbits  $\{\alpha\}$ ,  $\Delta_1$ ,  $\Delta_2$  of lengths 1, 891 and 1408 respectively.
- The self-dual **symmetric 1-designs**  $\mathcal{D}_i$  and associated **binary codes**  $C_i$  are constructed from the sets  $\Delta_1$ ,  $\{\alpha\} \cup \Delta_1$ ,  $\Delta_2$ ,  $\{\alpha\} \cup \Delta_2$ , and  $\Delta_1 \cup \Delta_2$ , respectively. We let  $\Omega = \{\alpha\} \cup \Delta_1 \cup \Delta_2$ .

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Let

$$S = \{|\Delta_1|, |\{\alpha\} \cup \Delta_1|, |\Delta_2|, |\{\alpha\} \cup \Delta_2|, |\Delta_1 \cup \Delta_2|\}.$$

Then

$$S = \{891, 892, 1408, 1409, 2299\}.$$

Then we have the following main result concerning  $\mathcal{D}_i$  and  $C_i$  for  $i \in S$

# Proposition 11

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- (i)  $\text{Aut}(\mathcal{D}_{891}) = \text{Aut}(\mathcal{D}_{892}) = \text{Aut}(\mathcal{D}_{1408}) = \text{Aut}(\mathcal{D}_{1409}) = \text{Aut}(\mathcal{C}_{892}) = \text{Aut}(\mathcal{C}_{1408}) = Co_2$ .
- (ii)  $\dim(\mathcal{C}_{892}) = 23$ ,  $\dim(\mathcal{C}_{1408}) = 22$ ,  
 $\mathcal{C}_{892} \supset \mathcal{C}_{1408}$  and  $Co_2$  acts irreducibly on  $\mathcal{C}_{1408}$ .
- (iii)  $\mathcal{C}_{891} = \mathcal{C}_{1409} = \mathcal{C}_{2299} = V_{2300}(GF(2))$ .
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## Proof of Proposition 11

- The proof of the theorem follows from a series of lemmas.
- In fact we will show that the codes  $C_{892}$  and  $C_{1408}$  are of types  $[2300, 23, 892]_2$  and  $[2300, 22, 1024]_2$  respectively.
- Furthermore

$$\begin{aligned}C_{892} &= \langle C_{1408}, j \rangle = C_{1408} \cup \{w + j : w \in C_{1408}\} \\ &= C_{1408} \oplus \langle j \rangle,\end{aligned}$$

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## Proof of Proposition 11 Cont.

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- we show that the code  $C_{1408}$  is the 22 dimensional irreducible representation of  $Co_2$  over  $GF(2)$  contained in the **23-dimensional decomposable**  $C_{892}$  (we called  $L$ )
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The weight distribution of  $C_{892} = L$

$l$	$A_l =  W_l $
0, 2300	1
892, 1408	2300
1024, 1276	46575
1100, 1200	476928
1136, 1164	1619200
1148, 1152	2049300

Action of  $Co_2$  on  $C_{892} = L$

Stabilizer (two copies)	Orbit length (two copies)
$Co_2$	1
$U_6(2) : 2$	2300
$2^{10} : M_{22} : 2$	46575
$HS : 2$	476928
$U_4(3) \cdot D_8$	1619200
$2_+^{1+8} : S_8$ non-maximal	2049300

The weight distribution of  $C_{1408} = N$

$l$	$A_l$
0	1
1024	46575
1136	1619200
1152	2049300
1200	476928
1408	2300

Stabilizer of a word  $w_I \in C_{1408}$

$I$	$(Co_2)_{w_I}$	Maximality
1024	$2^{10}:M_{22}:2$	Yes
1136	$U_4(3).D_8$	Yes
1152	$2_+^{1+8} : S_8$	No
1200	$HS:2$	Yes
1408	$U_6(2):2$	Yes

- The code  $C_{892}$  is self-orthogonal doubly-even, with minimum distance 892. It is a  $[2300, 23, 892]_2$  code.
- Its dual  $C_{892}^\perp$  is a  $[2300, 2277, 4]_2$  code.
- Moreover  $j \in C_{892}^\perp$  and  $j \in C_{892}$ .
- $C_{1408}$  is self-orthogonal doubly even, with minimum distance 1024. It is a  $[2300, 22, 1024]_2$  code.
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We should also mention that computation with Magma shows the codes over some other primes, in particular,  $p = 3$  are of some interest. In a separate paper we plan to deal with the ternary codes invariant under  $Co_2$  [31].

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