Families of polynomials appearing in the study of infinite dimensional Lie algebras.

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Abstract

Many types of polynomials arise naturally in the representation theory of Lie groups and Lie algebras. We will show how families of (nonclassical) orthogonal polynomials such as ultra spherical, Pollaczek, associated Legendre, associated Jacobi, and Cheybshev polynomials appear. Such polynomials arise when describing the universal central extension of particular families of Krichever-Novikov algebras and their automorphism groups. The associated Jacobi polynomials of Ismail and Wimp satisfy certain fourth order linear differential equations that also are related to the work of Kaneko and Zagier on supersingular j-invariants and Atkins polynomials. We will describe this family of differential equations. This is joint work with V. Futorny, J. Tirao, and R. Lu, X. Guo and K. Zhao.

The Universal Central Extension.

An *extension* of a Lie algebra L is a short exact sequence of Lie algebras

$$0 \longrightarrow I \xrightarrow{f} L' \xrightarrow{g} L \longrightarrow 0.$$

A *homomorphism* from one extension $g : L' \to L$ to another extension $g' : L'' \to L$ is a Lie algebra homomorphism $h : L' \to L''$ such that $g' \circ h = g$.

A *central extension* of a Lie algebra L is an extension such that $\ker g \subset Z(L')$ where $Z(L')$ is the center of the Lie algebra L'.

A central extension $u : \hat{L} \to L$ is called a *universal central extension* if there exists a unique homomorphism from $u : \hat{L} \to L$ to any other central extension $g : L' \to L$.

Experience has shown that universal central extensions have a "richer' representation theory than just the algebra L itself.

There is a somewhat abstract construction for the universal central extension exists if the Lie algebra L is perfect meaning $[L, L] = L$. Note the three dimensional Heisenberg Lie algebra is a non-perfect a central extension but it doesn't satisfy the uniqueness property for the map u above.

Universal Central Extension of $\mathfrak{g} \otimes R$

- R be a commutative algebra defined over $\mathbb{C}$ and $\mathfrak{g}$ is a finite dimensional simple Lie algebra defined over $\mathbb{C}$.
- $\Omega_R^1 = F/K$ is the module of Kähler differentials: it is the left R-module with action $f(g \otimes h) = fg \otimes h$ for $f, g, h \in R$ and K is the submodule generated by $1 \otimes fg - f \otimes g - g \otimes f$.
- The element $f \otimes g + K$ is traditionally denoted by fdg. The canonical map $d : R \to \Omega_R^1$ is given by $df = 1 \otimes f + K$. The *exact differentials* are the elements of the subspace dR. The coset of fdg modulo dR is denoted by $\overline{fdg}$.
- $\hat{\mathfrak{g}} = (\mathfrak{g} \otimes R) \oplus \Omega_R^1/dR$ is the Lie algebra with bracket given by

$$[x \otimes f, y \otimes g] = [xy] \otimes fg + (x, y)\overline{fdg}, \;\; [x \otimes f, \omega] = 0, \;\; [\omega, \omega'] = 0,$$

where $x, y \in \mathfrak{g}$, and $\omega, \omega' \in \Omega_R^1/dR$ and (x, y) denotes the Killing form on $\mathfrak{g}$.

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Lie algebra 2-cocycles

A Lie algebra 2-cocycle for a Lie algebra L defined over the complex numbers is a bilinear map $\psi : L \times L \to \mathbb{C}$ satisfying the two conditions

1. $\psi(a, b) = -\psi(b, a)$, for all $a, b \in L$,
2. $\psi([a, b], c) + \psi([b, c], a) + \psi([c, a], b) = 0$, for all $a, b, c \in L$.

The assignment $\psi : (\mathfrak{g} \otimes R) \times (\mathfrak{g} \otimes R) \to \mathbb{C}$ given by

$$\psi(x \otimes f, y \otimes g) := (x, y)\overline{f\,dg} \tag{3.1}$$

is a 2-cocycle on the Lie algebra $\mathfrak{g} \otimes R$. Here one finds a basis for Ω_R/dR and with respect to that basis the coefficients of the basis satisfy the conditions for being a 2-cocycle.

Main Problem: Given a basis of R find a basis of Ω_R/dR and then describe explicitly the 2-cocycle in terms of these basis. Many families of known polynomials naturally appear.

Examples for basis of Ω_R/dR

1. If $R = \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, then the universal central extension of $\mathfrak{g} \otimes R$ for $n \geq 1$ is called a **toroidal** Lie algebra. In this case Ω_R/dR has a countable but infinite basis

$$\overline{t_1^{m_1} t_2^{m_2} \cdots t_i^{-1} \cdots t_n^{m_n} dt_i}$$

$m_j \in \mathbb{Z}$ and $1 \leq i \leq n$ (S. Eswara Rao and R. Moody 1994).

2. Let $a_1, \ldots, a_n$ be distinct complex numbers. If $R = \mathbb{C}[t^{\pm 1}, (t-a_1)^{-1} \ldots, (t-a_n)^{-1}]$, then Ω_R/dR has a basis

$$\overline{t_i^{-1} dt}, \ \overline{(t-a_1)^{-1} dt}, \ldots, \overline{(t-a_n)^{-1} dt}.$$

This is a result due to M. Bremner (1994). Here $\mathfrak{g} \otimes R$ is an example of an $n+2$-point algebra.

Multipoint Krichever-Novikov algebras

Let Σ be a compact Riemann surface of genus g and A a finite number of $n \geq 2$ points in Σ. R will be the ring of meromorphic functions on on Σ with poles only allowed at A. There are at least two families of **Krichever-Novikov** algebras arising from the study of integrable systems: central extensions of

- the current algebra type $\mathfrak{g} \otimes R$,
- the algebra of derivations $\mathrm{Der}(R)$.

All of the algebras we will talk about today are examples (of central extensions) of Krichever-Novikov algebras.

Three useful results

1. $$\dim_{\mathbb{C}} \Omega_R/dR = 2g + n - 1$$
   (by a result of Grothendieck).
2. $\mathrm{Der}(R)$ are infinite dimensional **simple** Lie algebras (and thus perfect) due to D. A. Jordan (1986).
3. Dimension of the center of the universal central extension of $\mathrm{Der}(R)$ is also $2g + n - 1$ due to S. Skryabin (2004).

Examples: 4-point algebras

Let $a \in \mathbb{C}$ with $a \neq 0, 1$. If

$$R = \mathbb{C}[t^{\pm 1}, (t-1)^{-1}, (t-a)^{-1}],$$

then $R \cong S$ where $S = \mathbb{C}[t, t^{-1}, u \,|\, u^2 = t^2 - 2bt + 1]$ for some complex number $b \neq \pm 1$. In this case we have Ω_S/dS has a basis

$$\omega_0 := \overline{t^{-1}\,dt}, \quad \omega_+ := \overline{t^{-1}u\,dt}, \quad \omega_- := \overline{t^{-2}u\,dt}.$$

Proposition (Bremner, 1995)

For $i \in \mathbb{Z}$, $j \in \mathbb{Z}$, one has

$$\overline{t^{i-\frac{1}{2}}ud(t^j)} = \begin{cases} jQ_{i+j-\frac{3}{2}}(b)(b\omega_+ + \omega_-) & \text{for } i+j \geq 3/2, \\ j\omega_\pm & \text{for } i+j = \pm 1/2, \\ jQ_{-i-j-\frac{3}{2}}(b)(\omega_+ + b\omega_-) & \text{for } i+j \leq -3/2, \end{cases} \tag{6.1}$$

where

$$Q_k(b) := -\frac{P_{k+2}(b)}{b^2-1}$$

and $P_k(b)$ are the ultra spherical polynomials with generating function $(z^2 - 2bz + 1)^{-1/2}$.

Examples: Elliptic algebras

Let

$$R = \mathbb{C}[t, t^{-1}, u \,|\, u^2 = t^3 - 2bt^2 + t]$$

for some complex number $b \neq \pm 1$. This is the coordinate ring of the an elliptic curve with a finite number of points removed. In this case we have Ω_R/dR has a basis

$$\omega_0 := \overline{t^{-1}\,dt}, \quad \omega_+ := \overline{t^{-1}u\,dt}, \quad \omega_- := \overline{t^{-2}u\,dt}.$$

The four parameter **Pollaczek** polynomials $P_k(b) = P_k^{\lambda}(b; \alpha, \beta, \gamma)$ satisfy the recursion relation

$$(k+\gamma)P_k(b) = 2[(k+\lambda+\alpha+\gamma-1)b+\beta)]P_{k-1}(b) - (k+2\lambda+\gamma-2)P_{k-2}(b). \quad (7.1)$$

Proposition (Bremner, 1995)

For $i, j \in \mathbb{Z}$, one has

$$\overline{t^{i-1}ud(t^j)} = j(p_{|i+j|}(b)\omega_- + q_{|i+j|}(b)\omega_-) \quad (7.2)$$

where p_k and q_k are the the Pollaczek polynomials $P_k^{-1/2}(b; 0, -1, 1/2)$ with initial conditions $p_0(b) = 1$, $p_1(b) = 0$, respectively $q_0(b) = 0$, $q_1(b) = 1$.

DJKM algebras

Date, Jimbo, Kashiwara and Miwa studied in 1986 integrable systems arising from Landau-Lifshitz differential equation:

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx} + \mathbf{S} \times \mathbf{J}\mathbf{S} \tag{8.1}$$

where

$$\mathbf{S} = (S_1, S_2, S_3), \quad S_1^2 + S_2^2 + S_3^2 = 1, \quad \mathbf{J} = \begin{pmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{pmatrix}$$

J_i are constants. The authors introduced an infinite-dimensional Lie algebra which is a one dimensional central extension of

$$\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}, u | u^2 = (t^2 - b^2)(t^2 - c^2)] \tag{8.2}$$

where $b \neq \pm c$ are complex constants and $\mathfrak{g}$ is a simple finite dimensional Lie algebra. This Lie algebra acts on the solutions of the Landau-Lifshitz equation as infinitesimal Bäcklund transformations. These are the DJKM algebras and they are a particular type of KN-algebra.

Consider the ring

$$R = \mathbb{C}[t, t^{-1}, u | u^2 = p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0].$$

Theorem (M. Bremner)

Let R be as above. The set

$$\{\overline{t^{-1}\,dt}, \overline{t^{-1}u\,dt}, \ldots, \overline{t^{-n}u\,dt}\}$$

forms a basis of Ω^1_R/dR (omitting $\overline{t^{-n}u\ dt}$ if $a_0 = 0$).

Lemma (C.-Futorny)

If $u^m = p(t)$ and $R = \mathbb{C}[t, t^{-1}, u | u^m = p(t)]$, then in Ω^1_R/dR, one has

$$((m+1)n + im)t^{n+i-1}u\,dt \equiv -\sum_{j=0}^{n-1}((m+1)j + mi)a_j t^{i+j-1}u\,dt \mod dR \tag{9.1}$$

In the Date-Jimbo-Miwa-Kashiwara setting we have $p(t) = (t^2 - a^2)(t^2 - b^2)$ whereby after a change of variables we may assume $p(t) = t^4 - 2ct^2 + 1$ so that the recursion relation looks like

$$(6 + 2k)\overline{t^k u\,dt} = -2(k-3)\overline{t^{k-4} u\,dt} + 4kc\overline{t^{k-2} u\,dt}, \tag{9.2}$$

Let $P_k := P_k(c)$ be the polynomial in c satisfy the recursion relation

$$(6 + 2k)P_k(c) = 4kcP_{k-2}(c) - 2(k-3)P_{k-4}(c)$$

for $k \geq 0$.

Problem: What are the P_k? Are they classical orthogonal polynomials of some sort?

We follow ideas found in Arken's book on "Mathematical Methods for Physicists". Form the generating function

$$P(c,z) := \sum_{k\geq -4} P_k(c) z^{k+4} = \sum_{k\geq 0} P_{k-4}(c) z^k.$$

Then the recursion relation implies that $P(c,z)$ must satisfy the differential equation

$$\frac{d}{dz}P(c,z) - \frac{3z^4-4cz^2+1}{z^5-2cz^3+z}P(c,z) = \frac{2\left(P_{-1}+cP_{-3}\right)z^3 + P_{-2}z^2 + (4cz^2-1)P_{-4}}{z^5-2cz^3+z} \tag{10.1}$$

where $P_{-1}, P_{-2}, P_{-3}, P_{-4}$ are arbitrary constants. This has integrating factor

$$\begin{aligned}\mu(z) &= \exp\int\left(\frac{-2\left(z^3-cz\right)}{1-2cz^2+z^4} - \frac{1}{z}\right)dz \\ &= \exp(-\frac{1}{2}\ln(1-2cz^2+z^4) - \ln(z)) = \frac{1}{z\sqrt{1-2cz^2+z^4}}.\end{aligned}$$

If we take initial conditions $P_{-3}(c) = P_{-2}(c) = P_{-1}(c) = 0$ and $P_{-4}(c) = 1$ then we arrive at a generating function

$$P_{-4}(c,z) := \sum_{k \geq -4} P_{-4,k}(c) z^{k+4} = \sum_{k \geq 0} P_{-4,k-4}(c) z^k,$$

defined in terms of an elliptic integral

$$P_{-4}(c,z) = z\sqrt{1 - 2cz^2 + z^4} \int \frac{4cz^2 - 1}{z^2(z^4 - 2cz^2 + 1)^{3/2}}\, dz.$$

If we take initial conditions $P_{-4}(c) = P_{-3}(c) = P_{-1}(c) = 0$ and $P_{-2}(c) = 1$, we arrive at a generating function defined in terms of another elliptic integral:

$$P_{-2}(c,z) = z\sqrt{1 - 2cz^2 + z^4} \int \frac{1}{(z^4 - 2cz^2 + 1)^{3/2}}\, dz.$$

If we take $P_{-1}(c) = 1$, and $P_{-2}(c) = P_{-3}(c) = P_{-4}(c) = 0$ and set $P_{-1}(c,z) = \sum_{n\geq 0} P_{-1,n-4}z^n$, then

$$P_{-1}(c,z) = \frac{1}{c^2-1}\left(cz - z^3 - cz + c^2z^3 - \sum_{k=2}^{\infty} cQ_n^{(-1/2)}(c)z^{2n+1}\right),$$

where $Q_n^{(-1/2)}(c)$ is the n-th Gegenbauer polynomial. The $Q_n^{(-1/2)}(c)$ are known to satisfy the second order differential equation:

$$(1-c^2)\frac{d^2}{d^2c}Q_n^{(-1/2)}(c) + n(n-1)Q_n^{(-1/2)}(c) = 0$$

so that the $P_{-1,k} := P_{-1,k}(c)$ satisfy the second order differential equation

$$\boxed{(c^4-c^2)\frac{d^2}{d^2c}P_{-1,2n-3} + 2c(c^2+1)\frac{d}{dc}P_{-1,2n-3} + (-c^2n(n-1)-2)P_{-1,2n-3} = 0} \tag{10.2}$$

for $n \geq 2$.

Next we consider the initial conditions $P_{-1}(c) = 0 = P_{-2}(c) = P_{-4}(c) = 0$ with $P_{-3}(c) = 1$ and set

$$P_{-3}(c,z) = \sum_{n\geq 0} P_{-3,n-4}(c) z^n = \frac{1}{c^2-1}\left(c^2 z - cz^3 - z + cz^3 - \sum_{k=2}^{\infty} Q_n^{(-1/2)}(c) z^{2n+1}\right),$$

where $Q_n^{(-1/2)}(c)$ is the n-th Gegenbauer or ultraspherical polynomial. Hence

$$\boxed{(c^2-1)\frac{d^2}{d^2c}P_{-3,2n-3} + 4c\frac{d}{dc}P_{-3,2n-3} - (n+1)(n-2)P_{-3,2n-3} = 0} \tag{10.3}$$

for $n \geq 2$ and $P_{-1,2n-3} = cP_{-3,2n-3}$ for $n \geq 2$.

The fourth order linear differential equations

If we take initial conditions $P_{-4}(c) = P_{-3}(c) = P_{-1}(c) = 0$ and $P_{-2}(c) = 1$, recall that we arrive at a generating function defined in terms of another elliptic integral: From now on we are going to reindex the polynomials $P_{-4,n}$:

$$P_{-4}(c,z) = z\sqrt{1-2cz^2+z^4}\int \frac{4cz^2-1}{z^2(z^4-2cz^2+1)^{3/2}}\,dz = \sum_{n=0}^{\infty} P_{-4,n}(c)z^n$$

$$= 1 + z^4 + \frac{4c}{5}z^6 + \frac{1}{35}\left(32c^2-5\right)z^8 + \frac{16}{105}c\left(8c^2-3\right)z^{10}$$

$$-\frac{\left(2048c^4-1248c^2+75\right)}{1155}z^{12} + O(z^{14})$$

This means that now $P_{-4,0}(c) = 1$, $P_{-4,1}(c) = P_{-4,2}(c) = P_{-4.3}(c) = 0$. The first few nonzero polynomials in c are

$$P_{-4,4}(c) = 1, \quad P_{-4,6} = \frac{4c}{5}, \quad P_{-2,8} = \frac{32c^2 - 5}{35}$$

$$P_{-2,10} = \frac{16}{105} c \left(8c^2 - 3\right), \quad P_{-2,12} = -\frac{\left(2048c^4 - 1248c^2 + 75\right)}{1155}$$

and the coefficients $P_{-4,n}(c)$ of the generating function satisfy

$$\boxed{\begin{aligned} &16(c^2-1)^2 P_n^{(iv)} + 160c(c^2-1)P_n''' - 8(c^2(n^2-4n-46) - n^2 + 4n + 22)P_n'' \\ &\quad - 24c(n^2-4n-6)P_n' + (n-4)^2 n^2 P_n = 0. \end{aligned}}$$

We reindex the polynomials $P_{-2,n}$:

$$P_{-2}(c,z) = z\sqrt{1-2cz^2+z^4}\int \frac{1}{(z^4-2cz^2+1)^{3/2}} = \sum_{n=0}^{\infty} P_{-2,n}(c)z^n.$$

This means that now $P_{-2,2}(c)=1$, $P_{-2,3}(c)=P_{-2,1}(c)=P_{-2,0}(c)=0$ and the first few nonzero polynomials in c are

$$P_{-2,2}(c)=1, \quad P_{-2,6}=1/5, \quad P_{-2,8}=8c/35,$$

$$P_{-2,10},=(-7+32c^2)/105, \quad P_{-2,12}=8c(-29+64c^2)/1155.$$

so that

$$P_{-2}(c,z) = z^2+\frac{1}{5}z^6+\frac{8c}{35}z^8+\frac{32c^2-7}{105}z^{10}+\frac{8c(64c^2-29)}{1155}z^{12}+O(z^{14}).$$

After a very similar lengthy analysis we arrive at the following result: The polynomials $P_n = P_{-2,n}(c)$ are solutions to the family of ODE's

$$\boxed{\begin{aligned}&16(c^2-1)^2P_n^{(iv)}+160c(c^2-1)P_n'''-8(c^2(n^2-4n-42)-n^2+4n+18)P_n''\\&\quad-24c(n^2-4n-2)P_n'+(n-6)(n-2)^2(n+2)P_n=0.\end{aligned}}$$

Theorem (Favard)

Let p_n be a family of polynomials of degree n satisfying the following recursion

$$xp_n = a_{n+1}p_{n+1} + b_n p_n + c_{n-1}p_{n-1},$$

for some complex numbers a_i, b_j, c_k. Then $\{p_n, n \geq 0\}$ is an orthonormal family of polynomials with respect to some moment functional if $c_{n-1} = \bar{a}_n$ for all n.

This means that with respect to some moment functional $\mathcal{L}$ one has $\mathcal{L}[p_m(x)p_n(x)] = 0$ for $m \neq n$.

The polynomials P_n above do not satisfy the hypothesis of this theorem, but scalar multiples of them do. Hence the P_n are orthogonal polynomials.

A family of orthogonal polynomials q_n of degree n is said to be **classical** if they are eigenfunctions of the differential operator

$$D = (ax^2 + bx + c)\left(\frac{d}{dx}\right)^2 + (ex + f)\frac{d}{dx}, \tag{10.4}$$

with eigenvalues that are polynomials of degree 2 in n. In other words they satisfy equations of the following type

$$(ax^2 + bx + c)\left(\frac{d}{dx}\right)^2 q_n + (ex + f)\frac{d}{dx}q_n = (an(n-1) + bn + c + en + f)q_n, \tag{10.5}$$

for some constants a, b, c, d, e, f and for all $n \geq 0$.
The families of polynomials $P_{-4,n}$ and $P_{-2,n}$ for $n \geq 0$ do not satisfy such differential equations for all n so they are non-classical. The $P_{-4,n}$ are specializations of the associated Jacobi polynomials studied by J. Bustoz, M. Ismail (1982) and J. Wimp (1987). The orthogonal polynomials $P_{-2,n}$ and its corresponding fourth order linear differential equation appear to be new.

Set $\omega_0 := \overline{t^{-1}\,dt}, \omega_{-1} := \overline{t^{-1}u\,dt}, \omega_{-2} := \overline{t^{-2}u\,dt}, \omega_{-3}, := \overline{t^{-3}u\,dt}, \omega_{-4} := \overline{t^{-4}u\,dt}$

Theorem (C.-Futorny)

Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra over the complex numbers with the Killing form $(\,|\,)$ and define $\psi_{ij}(c) \in \Omega^1_R/dR$ by

$$\psi_{ij}(c) = \begin{cases} \omega_{i+j-2} & \text{for} \quad i+j = 1, 0, -1, -2 \\ P_{-3,i+j-2}(c)(\omega_{-3} + c\omega_{-1}) & \text{for} \quad i+j = 2n-1 \geq 3, \;\; n \in \mathbb{Z}, \\ P_{-3,i+j-2}(c)(c\omega_{-3} + \omega_{-1}) & \text{for} \quad i+j = -2n+1 \leq -3, n \in \mathbb{Z}, \\ P_{-4,|i+j|-2}(c)\omega_{-4} + P_{-2,|i+j|-2}(c)\omega_{-2} & \text{for} \quad |i+j| = 2n \geq 2, n \in \mathbb{Z}. \end{cases} \tag{10.6}$$

The universal central extension of the Date-Jimbo-Kashiwara-Miwa algebra is the $\mathbb{Z}_2$-graded Lie algebra

$$\widehat{\mathfrak{g}} = (\mathfrak{g} \otimes R) \oplus \mathbb{C}\omega_{-4} \oplus \mathbb{C}\omega_{-3} \oplus \mathbb{C}\omega_{-2} \oplus \mathbb{C}\omega_{-1} \oplus \mathbb{C}\omega_0$$

with bracket

$$\begin{aligned} [x \otimes t^i, y \otimes t^j] &= [x,y] \otimes t^{i+j} + \delta_{i+j,0}\, j(x,y)\omega_0, \\ [x \otimes t^{i-1}u, y \otimes t^{j-1}u] &= [x,y] \otimes (t^{i+j+2} - 2ct^{i+j} + t^{i+j-2}) \\ &\quad + \left(\delta_{i+j,-2}(j+1) - 2cj\delta_{i+j,0} + (j-1)\delta_{i+j,2}\right)(x,y)\omega_0, \\ [x \otimes t^{i-1}u, y \otimes t^j] &= [x,y]u \otimes t^{i+j-1} + j(x,y)\psi_{ij}(c). \end{aligned}$$

Recent work: Universal Central Extension of hyperelliptic current algebras

Let

$$p(t) = (t-\alpha_1)\cdots(t-\alpha_n) = \sum_{i=0}^{n} a_i t^i$$

where the α_i are distinct complex numbers and fix

$$R = \mathbb{C}[t, t^{-1}, u \,|\, u^2 = p(t)]$$

so that R is a regular ring. The $\mathfrak{g} \otimes R$ is a **hyperelliptic** Lie algebra. We let $P_{k,i} := P_{k,i}(a_0, \ldots, a_{n-1})$, $k \geq -n$, $-n \leq i \leq -1$ be the polynomials in the a_i satisfying the recursion relations

$$(2k+n+2)P_{k,i} = -\sum_{j=0}^{n-1}(3j+2k-2n+2)a_j P_{k-n+j,i} \tag{10.7}$$

Then using Faá de Bruno's formula we get

$$P_n(a_0, \ldots, a_{n-1}) = \frac{1}{n!}\sum_{l=0}^{n}\frac{(-1)^l(2l+1)!!}{2^l}B_{n,l}(a_{n-1}, 2a_{n-2}, \ldots, (n-l+1)!a_{l-1}). \tag{10.8}$$

where $B_{n,l}$ are Bell polynomials (C, 2015). (Bell's grave is in Watsonville, California).

Recent work: Universal Central Extension of Der(R).

There is an abstract description of the universal central extension of Der(R) for R a regular ring due to S. Skryabin - (regular means it has a finite global dimension or its local rings at all prime ideals are all regular local rings - the minimal number of generators of its maximal ideal is equal to its Krull dimension.)

In any case when $R = \mathbb{C}[t, t^{-1}, u \mid u^2 = p(t)]$ we have a description of the 2-cocyles in terms of the polynomials $P_n(a_0, \dots, a_{n-1})$ above. It also involves the formula of Faá de Bruno and Bell polynomials. When

$$p(t) = t^{2n} - 2a_n t^n + 1$$

for $n \geq 2$ fixed $a_n \neq \pm 1$, one obtains that the P_n are associated Legendre polynomials.

Automorphism group of Der(R): $R = \mathbb{C}[t, (t-a_1)^{-1}, \ldots, (t-a_n)^{-1}]$.

The automorphism group of Der(R) for R when R is a regular ring is completely determined by the automorphism group of R (by another result of S. Skryabin). This is a very hard problem in general for R the ring of meromorphic functions on a Riemann surface with a finite number of fixed poles.

One approach to determining the automorphism group of R is to first determine the group of units of R and then realize that units have to be mapped to units. We used this approach to prove the following result.

Theorem (C., Xiangqian Guo, Rencai Lu, Kaiming Zhao)

Let $R = \mathbb{C}[t, (t-a_1)^{-1}, \ldots, (t-a_n)^{-1}]$. The automorphism group of Der(R), where a_i are distinct is isomorphic to one of the following groups studied by Klein: S_4, A_4, A_5, C_n, D_n and all these groups appear as automorphism groups.

Automorphism group of Der(R): $R = \mathbb{C}[t, t^{-1}, u \mid u^2 = p(t)]$

An element $a_n + b_n u$ in the group of units of $R = \mathbb{C}[t, t^{-1}, u \mid u^2 = p(t)]$ must satisfy ' the polynomial Pell equation

$$a_n^2 - p(t) b_n^2 = t^k$$

for some $k \in \mathbb{Z}$. Remark: If $p(t) = t^2 - 1$ and $k = 0$, then it is known that $a_n(t) = T_n(t)$ and $b_n(t) = U_n(t)$ are the Chebyshev polynomials.

Using the description of the group of units of R we get

Theorem (C., Xiangqian Guo, Rencai Lu, Kaiming Zhao)

The automorphism group of $R = \mathbb{C}[t, t^{-1}, u \mid u^2 = t(t - a_1) \cdots (t - a_{2n})]$, *is of type* $\mathbb{Z}_k \times \mathbb{Z}_2$ *or* $D_k \times \mathbb{Z}_2$, $k \geq 2$.

Automorphism group of Der(R):

Last remark: If $p(t) = t^4 - 2\beta t^2 + 1$, $\beta \neq 1$, then one can form the generating function for a_n and b_n and use it to show that the b_n satisfy the second order Fuchsian differential equation of the form

$$P(t)y'' + Q_n(t)y' + R_n(t)y = 0$$

where

$$\begin{aligned}
P(t) &= t\left(t^2+1\right)\left(t^4 - 2\beta t^2 + 1\right), \\
Q_n(t) &= -\left((2n-3)t^6 + t^4(-4\beta n + 2n - 5) + t^2(4\beta - 4\beta n + 2n + 3) + 2n + 1\right) \\
&= -2(n-1)\left(t^2+1\right)\left(t^4 - 2\beta t^2 + 1\right) + t^6 + (4\beta+3)t^4 - 5t^2 - 3, \\
R_n(t) &= -2n\left(2t^5 + t^3(\beta + (\beta+1)n + 5) + t(-\beta + (\beta+1)n + 1)\right).
\end{aligned}$$

We found a fourth order linear differential equation that the a_n satisfy but the differential equation is too messy to put into a paper.