$\boxed{P_1}$

Selection of Problems

1. Show that any intersection of right topologizing filters on a ring R is a right topologizing filter on R.

2. Let A be a nonempty subset of $\mathrm{Fil}\, R_R$. Show that
$$\eta(A) = \{K \leq R_R : K \supseteq \bigcap_{i=1}^{n} r_i^{-1}A_i,\ A_i \in A,\ r_i \in R,\ 1 \leq i \leq n\}$$
is a right topologizing filter on R and is the smallest containing A.

3. Let A, B be right ideals of R. Prove that the following assertions are equivalent:
(i) $\eta(A) \subseteq \eta(B)$;
(ii) R/A is a homomorphic image of a submodule of $\prod_{i=1}^{n} R/B$.

Answers

1. This is routine.

2. Follows from the following two identities (check these!)
   - $r^{-1}(t^{-1}A) = (tr)^{-1}A$ for all right ideals A of R and $r, t \in R$.
   - $r^{-1}\left(\bigcap_{\delta\in\Delta} K_\delta\right) = \bigcap_{\delta\in\Delta} r^{-1}K_\delta$ for any family $\{K_\delta : \delta \in \Delta\}$ of right ideals in R, and $r \in R$.

P2

3. <u>(i) $\Rightarrow$ (ii)</u> $\eta(A) \subseteq \eta(B)$ implies $A \in \eta(B)$, so $A \supseteq \bigcap_{i=1}^{n} r_i^{-1}B$ for suitable $r_i \in R$, $1 \le i \le n$.

Put $A' = \bigcap_{i=1}^{n} r_i^{-1}B$. Check that the map

$$R/A' = R\Big/\bigcap_{i=1}^{n} r_i^{-1}B \longrightarrow \prod_{i=1}^{n} R/B$$

$$r + \bigcap_{i=1}^{n} r_i^{-1}B \longmapsto (r_1 r + B, r_2 r + B, \ldots, r_n r + B)$$

is a well-defined, one-to-one R-module homomorphism. Thus R/A' is isomorphic to a submodule of $\prod_{i=1}^{n} R/B$.

Since $A \supseteq A'$, there is a canonical epimorphism from R/A' onto R/A. This establishes (ii).

<u>(ii) $\Rightarrow$ (i)</u> Let L be a submodule of $\prod_{i=1}^{n} R/B$ and $\pi : L \longrightarrow R/A$ an epimorphism.

Pick $x \in L$ such that $\pi(x) = 1_R + A$. Since $x \in L \subseteq \prod_{i=1}^{n} R/B$, $x = (r_1 + B, \ldots, r_n + B)$ for suitable $r_i \in R$, $1 \le i \le n$. Note that $xt = (r_1 + B, \ldots, r_n + B)t = 0$ iff $t \in \bigcap_{i=1}^{n} r_i^{-1}B$.

But $xt = 0 \Rightarrow \pi(x)t = \pi(xt) = 0 \Leftrightarrow (1_R + A)t = 0$
$\Leftrightarrow t \in A$.

This shows that $A \supseteq \bigcap_{i=1}^{n} r_i^{-1}B$, so $A \in \eta(B)$ and $\eta(A) \subseteq \eta(B)$.