Kac-Moody algebras, vertex operators and applications

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Outline

- Virasoro algebra
- Heisenberg/Weyl algebras
- Kac-Moody algebras
- Krichever-Novikov algebras
- DJKM (Date-Jimbo-Kashiwara-Miwa) algebra

Sophus Lie, 1870

M manifold, $G = \mathrm{Diff}(M)$ $\Rightarrow$ 1-parameter subgroup $\{\varphi_t, t \in \mathbb{R}\} \subset G$;
$\gamma_m(t) := \varphi_t(m),\ m \in M;\quad v(m) = \gamma'_m(0)$

$$\mathrm{Vect}(M) = \{v : m \mapsto v(m), m \in M\}$$

Example: $M = S^1$, $\mathrm{Vect}_{\mathbb{C}}(S^1)$, $A_n = e^{in\varphi}\frac{d}{d\varphi}$,

$$[A_n, A_m] = (m-n)A_{n+m} \quad (\textit{Witt algebra})$$

$$[A_n, A_m] = (m-n)A_{n+m} + \frac{n^3-n}{12}\delta_{n,-m}\, c$$

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(*Virasoro algebra*)

A Lie algebra $\mathfrak{g}$ over field k: a vector space over k + a bilinear map $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that:

$$[x, x] = 0, \forall x \in \mathfrak{g}$$

(*anticommutativiity*)

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$$

(*Jacobi identity*)

1. Let A be an associative algebra. Define on a vector space A a Lie structure by:

$$[a, b] = ab - ba.$$

hence every associative algebra gives rise to a Lie algebra.

Reciprocally, if $\mathfrak{g}$ is Lie algebra then one associates the universal enveloping algebra $U(\mathfrak{g})$, certain universal associative algebra which contains $\mathfrak{g}$

2. Let V be a vector space over k with basis $e_1, \dots, e_n$. The symmetric algebra $S(V)$ is spanned over field k by the elements $e_1^{k_1} \dots e_n^{k_n}$ with commutative multiplication.

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Let $\mathfrak{g}$ be a simple Lie algebra with basis $g_1, \ldots, g_n$ and the structure constants c_{ij}^k:

$$[g_i, g_j] = \sum_{ik} c_{ij}^k g_k$$

Then $S(\mathfrak{g})$ has a structure of a Lie algebra with the *Lie–Poisson bracket*

$$\{g_i, g_j\} = \sum_{k=1} c_{ij}^k g_k$$

+ product rule

(Hamiltonian equation of the motion)

3. $V, dimV = 2n$, $<,>$ symplectic form on V (nondegenerate, skew-symmetric),
x_i, y_i, $i = 1, \ldots, n$ symplectic coordinates.

$C^\infty(V, \mathbb{R})$ smooth functions on V,

Poisson bracket: (F.Berezin, 1967)

$$\{P, Q\} = \sum \frac{\partial P}{\partial y_j}\frac{\partial Q}{\partial x_i} - \frac{\partial P}{\partial x_i}\frac{\partial Q}{\partial y_j}$$

4. **Heisenberg algebra** H_n, $n \leq \infty$:

$$H_n = kz \oplus \sum_{i=-n, i\neq 0}^{n} kx_i,$$

$[x_i, x_j] = \delta_{i,-j} z$, $[z, x_i] = 0$.

Take $A_n = U(H_n)/(z - a)$ for any $a \in k.$, $a \neq 0$. Then A_n is n-th **Weyl** algebra ($a = 1$).

Origin. System of quantum mechanics: Hilbert space + Hamiltonian operator, e.g.
$L^2(\mathbb{R})$, $T = \frac{d^2}{dt^2} - x^2$ (quantum harmonic oscil.);
Problem: find eigenvalues of T. To solve use

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Weyl algebras

The n-th Weyl algebra $A_n = A_n(\mathbb{k})$:
generators: x_i, ∂_i, $i = 1, \ldots, n$
relations

$$x_i x_j = x_j x_i, \;\; \partial_i \partial_j = \partial_j \partial_i,$$
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e.g. $\forall x, y \in A_1$, s.t. $xy - yx = 1 \Rightarrow \langle x, y \rangle = A_1$?

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Let $P_n = \mathbb{C}[x_1, \ldots, x_n]$. $\forall \phi \in End(P_n)$ define $\phi_i = \phi(x_i)$.

Then ϕ defines a polynomial function

$$F : \mathbb{C}^n \to \mathbb{C}^n,$$

$F(z_1, \ldots, z_n) = (\phi_1(z_1), \ldots, \phi_n(z_n))$.
Let

$$J_F(\bar{x}) = (\partial f_i / \partial x_j), \ \Delta(F) = det(J_F).$$

If F is invertible then $\Delta(F) \in \mathbb{C}^*$.

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⋄ **Jacobian Conjecture**: Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial function such that $\Delta(F) \in \mathbb{C}^*$. Then F is invertible and the inverse is polynomial (Keller, 1939).

⋄ Open for $n > 2$

⋄ True for $n = 1$

⋄ Dixmier Problem $\Rightarrow$ Jacobian Conjecture (Bass, McConnel, Wright, 1982; Bavula, 2001)

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⋄ Case $n = 1$: $F(x)$ is polynomial and $\frac{dF}{dx} = a \neq 0$. Hence

$$F(x) = ax + b$$

and the inverse is

$$G = \frac{x - b}{a}$$

⋄ $F : \mathbb{C} \to \mathbb{C}$, $F(z) = e^z$. Then $J_F(z) = e^z \neq 0$; F is not injective, e^z is periodic with period $2\pi i$.

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Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra over $\mathbb{C}$, $\mathcal{A}$ a commutative associative algebra. Consider a tensor product of vector spaces $\mathfrak{g} \otimes \mathcal{A}$. Then $\mathfrak{g} \otimes \mathcal{A}$ becomes a Lie algebra with $[x \otimes a, y \otimes b] = [x, y] \otimes ab$, $x, y \in \mathfrak{g}$, $a, b \in \mathcal{A}$.

Denote $\Omega^1_{\mathcal{A}}$ the space of Kähler 1-forms - the quotient of the free $\mathcal{A}$-module with basis da, $a \in \mathcal{A}$, modulo the relations:
(1) $d(ab) = (da)b + a(db)$,
(2) $d(a + b) = da + db$ and
(3) $dk = 0$ for $a, b \in \mathcal{A}$, $k \in \mathbb{C}$.

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C.Kassel (mid 1990's): the universal central extension $\hat{\mathfrak{g}}$ of $L(\mathfrak{g}) = \mathfrak{g} \otimes \mathcal{A}$ is the vector space $L(\mathfrak{g}) \oplus \Omega^1_{\mathcal{A}}/d\mathcal{A}$,

$$0 \to \Omega^1_{\mathcal{A}}/d\mathcal{A} \to \hat{\mathfrak{g}} \to L(\mathfrak{g}) \to 0,$$

where

$$[x \otimes f, y \otimes g] := [xy] \otimes fg + (x, y)\overline{fdg}, \quad [x \otimes f, \omega] = 0$$

for $x, y \in \mathfrak{g}$, $f, g \in \mathcal{A}$, $\omega \in \Omega^1_{\mathcal{A}}/d\mathcal{A}$, and $(\ ,\ )$ denotes the Killing form on $\mathfrak{g}$.

Kac-Moody algebras

Cartan matrix $A = (a_{ij})$, $a_{ij} \leq 0$, $i \neq j$, $a_{ii} = 2$, $a_{ij} = 0 \Rightarrow a_{ji} = 0$ for all i, j, A is symmetrizable (there exists a diagonal invertible matrix D such that DA is symmetric) and DA is positive definite $\Rightarrow$

generators + Serre relations $\Rightarrow$ simple complex finite-dimensional Lie algebras.

V.Kac, R.Moody (1967): without "positive definite" $\Rightarrow$ *Kac-Moody algebras*;
If A is positive semidefinite ($\det(a_{ij}) = 0$, with positive principal minors) $\Rightarrow$ *Affine* Lie algebras

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Example

$\mathcal{A} = \mathbb{C}[t, t^{-1}]$, $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, with the Lie bracket

$$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{m+n}$$

(*loop* algebra).

◇ the universal central extension $\tilde{g} = \hat{\mathfrak{g}} \oplus \mathbb{C}c \oplus \mathbb{C}d$,

$$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{m+n} + n(x, y)\delta_{n+m,0}c,$$

$d : \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$ degree derivation, $d(x \otimes t^n) = n(x \otimes t^n)$, $d(c) = 0$ (*non-twisted affine* Kac-Moody algebra).

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Geometric interpretation:

View $\mathbb{C}[t, t^{-1}]$ as algebra of meromorphic functions on the Riemann sphere, holomorphic outside of $t = 0$ and $t = \infty$. Then $\hat{\mathfrak{g}}$ can be viewed as $\mathfrak{g}$-valued meromorphic functions on the Riemann sphere with possible poles only at $t = 0$ and $t = \infty$.

Generalise: take any compact Riemann surface X of arbitrary genus g and arbitrary set of points P where the poles are allowed. Then define $\mathfrak{g}(X, P) = \mathfrak{g} \otimes R$, where R the ring of meromorphic on X functions with poles in P:

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(Krichever-Novikov, Schlichenmaier, Sheinman, Bremner).

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Elliptic curve case:

Consider a compact complex algebraic curve.
(It can be represented as a quotient of the complex plane $\mathbb{C}$ by the lattice $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\lambda$ with positive imaginary part of λ.)

Take the ring $\mathcal{A}$ of meromorphic functions on Σ which are holomorphic outside of the set $\{0, \frac{1}{2}(1+\lambda)\}$.
Let $\wp$ be the Weierstrass function:

$$\wp(z) := z^{-2} + \sum_{0 \neq \omega \in \Lambda} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

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Then

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3,\ g_2 = 60 \sum_{0\neq\xi\in\Lambda} \xi^{-4},\ g_3 = 140 \sum_{0\neq\xi\in\Lambda} \xi^{-6}.$$

Theorem (Bremner)

$$\mathcal{A} \cong \mathbb{C}[t, t^{-1}, u | u^2 = t^3 - 2bt^2 + t]$$

where b is some constant determined by $m = \wp\left(\frac{1}{2}(1+\lambda)\right)$.

In our example: $\Omega^1_{\mathcal{A}}/d\mathcal{A}$ has a basis

$$\omega_0 := \overline{t^{-1}dt}, \quad \omega_- := \overline{t^{-2}u\,dt}, \quad \omega_+ := \overline{t^{-1}u\,dt}.$$

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$$\overline{t^{k-2}u\,dt} = p_k(b)\overline{t^{-1}u\,dt} + q_k(b)\overline{t^{-2}u\,dt}.$$

These are Pollaczek polynomials that satisfy the following recurrent relation:

$$(k+\gamma)p_k(b) = 2[(k+\lambda+\alpha+\gamma-1)b+\beta]p_{k-1}(b)-(k+2\lambda+\gamma-2)p_{k-2}(b)$$

for the parameters $\lambda = -1/2$, $\alpha = 0$, $\beta = -1$, $\gamma = 1/2$ together with the initial conditions

$$p_0(b) = 0, \quad p_1(b) = 1.$$

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$$(k+\gamma)p_k(b) = 2[(k+\lambda+\alpha+\gamma-1)b+\beta]p_{k-1}(b)-(k+2\lambda+\gamma-2)p_{k-2}(b)$$

for the parameters $\lambda=-1/2$, $\alpha=0$, $\beta=-1$, $\gamma=1/2$ together with the initial conditions

$$p_0(b)=0, \quad p_1(b)=1.$$

DJKM algebra

Let $p(t) = (t^2 - a^2)(t^2 - b^2) = t^4 - (a^2 + b^2)t^2 + (ab)^2$ with $a \neq \pm b$ and neither a nor b is zero.

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(B.Cox, V.F., 2010):

The universal central extension of the DJKM algebra is the $\mathbb{Z}_2$-graded Lie algebra

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}^0 \oplus \widehat{\mathfrak{g}}^1,$$

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after setting $c = (a^2 + b^2)/2$, so that $p(t) = t^4 - 2ct^2 + 1$.

Let $P_k := P_k(c)$ be the polynomial in c satisfying the recursion relation

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$$P(c,z) := \sum_{k\geq -4} P_k(c) z^{k+4} = \sum_{k\geq 0} P_{k-4}(c) z^k.$$

Then $P(c,z)$ satisfies the differential equation

$$\frac{d}{dz}P(c,z) - \frac{3z^4 - 4cz^2 + 1}{z^5 - 2cz^3 + z} P(c,z) =$$

$$\frac{2\left(P_{-1} + cP_{-3}\right) z^3 + P_{-2} z^2 + (4cz^2 - 1) P_{-4}}{z^5 - 2cz^3 + z}$$

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is defined in terms of an elliptic integral

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One way to interpret the right hand integral is to expand $(z^4 - 2cz^2 + 1)^{-3/2}$ as a Taylor series about $z = 0$ and then formally integrate term by term and multiply the result by the Taylor series of $z\sqrt{1 - 2cz^2 + z^4}$.

More precisely one integrates formally with zero constant term

$$\int (4c - z^{-2}) \sum_{n=0}^{\infty} Q_n^{(3/2)}(c) z^{2n}\, dz$$

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After multiplying this by

$$z\sqrt{1-2cz^2+z^4}=\sum_{n=0}^{\infty} Q_n^{(-1/2)}(c)z^{2n+1}$$

one arrives at the series $P_{-4}(c,z)$:

$$z\sqrt{1-2cz^2+z^4}\int \frac{4cz^2-1}{z^2(z^4-2cz^2+1)^{3/2}}\,dz=\sum_{n=0}^{\infty} P_{-4,n}(c)z^n$$

$$=1+z^4+\frac{4c}{5}z^6+\frac{1}{35}\left(32c^2-5\right)z^8+\frac{16}{105}c\left(8c^2-3\right)z^{10}$$
$$-\frac{(2048c^4-1248c^2+75)}{1155}z^{12}+O(z^{14})$$

The first nonzero polynomials in c are

$$P_{-4,4}(c) = 1, \quad P_{-4,6} = \frac{4c}{5}, \quad P_{-4,8} = \frac{32c^2 - 5}{35}$$

whose coefficients satisfy the following differential equation:

$$16(c^2-1)^2P_n^{(4)}+160c(c^2-1)P_n^{(3)}-a(n,c)P_n''$$

$$-24c(n^2-4n-2)P_n'+(n-6)(n-2)^2(n+2)P_n=0$$

Vertex realizations

Let $\dim \mathfrak{g} < \infty$, $\mathfrak{b}$ a Borel subalgebra, $\mathfrak{g} = LieG$, $\mathfrak{b} = LieB$

$\Rightarrow$ G acts transitively on the *flag variety* $X = G/B$ (smooth algebraic variety)

Take $B = B_-$. Action of N_+ on X leads to the decomposition of X into Shubert cells:

$X = \cup_{w \in W} C(w)$, $C(w) = B_+ w B_- / B_-$, $W = N(T)/T$ the Weyl group, $T = B_+/N_+$.

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The largest Shubert cell $\mathcal{U} = N_+$ ($w = 1$) can be identified with some affine space $\Rightarrow$ the ring of regular functions $\mathcal{O}_{\mathcal{U}}$ on $\mathcal{U}$ is a polynomial ring and $\mathfrak{g}$ has an embedding into some Weyl algebra (acts by certain differential operators).

Example Let $\mathfrak{g} = sl(2)$, standard basis e, f, h, $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$, $\mathfrak{b}_- = span\{f, h\} \Rightarrow G = SL_2(\mathbb{C})$ and $X = G/B_-$ can be identified with the projective line $\mathbb{P}^1$, $\mathcal{U} = \mathbb{A}^1$. Let x be a coordinate in $\mathcal{U}$, $\mathcal{O}_{\mathcal{U}} = \mathbb{C}[x] \Rightarrow$

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Let $\mathfrak{g} = sl(2)$,

$$e_n = e \otimes t^n,\ h_n = h \otimes t^n,\ f_n = f \otimes t^n, n \in \mathbb{Z}.$$

Using *natural Borel subalgebra* (which leads to "semi-infinite variety" by Feigin, Frenkel, Voronov) we obtain

$$e_n \mapsto \partial x_n,\quad h_n \mapsto -2\sum_{m\in\mathbb{Z}} x_m \partial x_{n+m},$$

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Theorem (Jakobsen-Kac, 85; Bernard-Felder, 90)

Formulas

$$f_n \mapsto x_n,$$

$$h_n \mapsto -(\lambda_n + 2\sum_{m\in\mathbb{Z}} x_{n+m}\partial x_m),$$

$$e_n \mapsto -(\sum_m \lambda_{m+n}\partial x_m + \sum_{m,k\in\mathbb{Z}} x_{n+m+k}\partial x_m \partial x_k)$$

define a representation of $\hat{\mathfrak{g}}$ *on a Fock space* $\mathbb{C}[x_m, m\in\mathbb{Z}]$ *with* $c = 0$, *where* $\lambda_m = \int_{S^1} z^m d\mu$ *for any finite measure* μ *on the unit circle*

Second free field realization

$$a(z)_- = \sum_{n<0} a_n z^{-n-1}, \quad a(z)_+ = \sum_{n\geq 0} a_n z^{-n-1}$$

$\Rightarrow$ *normal ordering*

$$: a(z)b(z) := a(z)_- b(z) + b(z)a_+(z)$$

Let

$$a_n = \begin{cases} x_n, & n < 0 \\ \partial x_n, & n \geq 0, \end{cases} \quad a_n^* = \begin{cases} x_{-n}, & n \leq 0 \\ -\partial x_{-n}, & n > 0, \end{cases}$$

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Theorem (Wakimoto, 86)

The formulas

$$c \mapsto K, \quad e(z) \mapsto a(z), \quad h(z) \mapsto -2 : a^*(z)a(z) :$$

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define the second free field realization *of the affine* $sl(2)$ *acting on the Fock space* $\mathbb{C}[x_n, n \in \mathbb{Z}]$.

B.Feigin, E.Frenkel (1990) - for any Affine Lie algebra.

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Vertex algebras

McKay-Thompson conjecture: $\exists M$ and a $\mathbb{Z}$-graded M-module V s.t.

$$\sum_{n\geq -1} (dim V_n) q^n = J(q),$$

$$J(q) = q^{-1} + 196884q + 21493760q^2 + \dots$$

R.Griess, 1980, largest sporadic group - *Monster* ("friendly giant"), $|M| \approx 10^{54}$, $M < AutB$, $dimB = 196883$, commutative, non-associative

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$\Rightarrow < 1, y_n, \partial/\partial y_n, Y_i, n, i \in 1/2\mathbb{Z}, n > 0 > \simeq \hat{s}l(2)$.

This is *principal* realization of $\hat{s}l(2)$. *Homogeneous* realization is due to I.Frenkel-Kac and Seagal.

Integrable representations $\Rightarrow$ character formula $\Rightarrow$ *Macdonald* identities for Dedekind η-function (Kac, Lepowsky-Wilson, Dyson,...), e.g. Jacobi identity:

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$$\Pi_{n\geq 1}(1-x^n y^n)(1-x^{n-1} y^n)(1-x^n y^{n-1}) = \sum_{j\in\mathbb{Z}} (-1)^j x^{1/2j(j+1)} y^{1/2j(j-1)}$$

Example(Lepowsky-Wilson). $W = \mathbb{C}[x_{1/2}, x_{3/2} \ldots,]$,

$$Y = exp(\sum_{n=1/2,3/2,\ldots} \frac{y_n}{n} x^n) exp(-2 \sum_{n=1/2,3/2,\ldots} \frac{\partial}{\partial y_n} x^{-n})$$

$\Rightarrow < 1, y_n, \partial/\partial y_n, Y_i, n, i \in 1/2\mathbb{Z}, n > 0 > \simeq \hat{sl}(2)$.
This is *principal* realization of $\hat{sl}(2)$. *Homogeneous* realization is due to I.Frenkel-Kac and Seagal.

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Vertex algebras

Chiral algebras in CFT (Belavin, Polyakov, Zamolodchikov); modular invarience (Witten); *vertex algebras* (Borcherds).

A vertex algebra is a vector space V with a distinguished vector $\mathbb{1}$ (vacuum vector) in V, an operator D (translation) on the space V, and a linear map Y (state-field correspondence)

$$Y(\cdot, z) : V \to (\mathrm{End}\, V)[[z, z^{-1}]],$$

$$a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$$

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following axioms hold:
(V1) For any $a, b \in V, \quad a_{(n)}b = 0$ for n sufficiently large;
(V2) $[D, Y(a, z)] = Y(D(a), z) = \frac{d}{dz} Y(a, z)$ for any $a \in V$;
(V3) $Y(\mathbb{1}, z) = Id_V\, z^0$;
(V4) $Y(a, z)\mathbb{1} \in V[[z]]$ and $Y(a, z)\mathbb{1}|_{z=0} = a$ for any $a \in V$ (self-replication);
(V5) Locality:

$$(z - w)^n\, [Y(a, z), Y(b, w)] = 0, \quad \text{for } n \text{ sufficiently large.}$$

A vertex algebra V is called a vertex operator algebra (VOA) if, in addition, V contains a vector ω (Virasoro element) such that
(V6) The components $L_n = \omega_{(n+1)}$ of the field

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} \omega_{(n)} z^{-n-1} = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$$

satisfy the Virasoro algebra relations:

$$[L_n, L_m] = (n-m)L_{n+m} + \delta_{n,-m} \frac{n^3 - n}{12} C_{vir},$$

where C_{vir} acts on V by scalar, called the *rank* of V.
(V7) $D = L_{-1}$;
(V8) Operator L_0 is diagonalizable on V.

Affine gl_N VOA.

. Let

$$\hat{gl}_N = \mathbb{C}[t, t^{-1}] \otimes gl_N \oplus \mathbb{C}c$$

with the standard Lie bracket.

Let $\hat{gl}_N = \hat{gl}_N^- \oplus \hat{gl}_N^0 \oplus \hat{gl}_N^+$ with $\hat{gl}_N^0 = 1 \otimes gl_N \oplus \mathbb{C}c$ and $\hat{gl}_N^{\pm 1} = t^{\pm 1}\mathbb{C}[t^{\pm 1}] \otimes gl_N$.

Let $\mathbb{C}\mathbb{1}$ be a 1-dimensional module for $\hat{gl}_N^0 \oplus \hat{gl}_N^+$ with the following action:

$$(\mathbb{C}[t] \otimes gl_N)\, \mathbb{1} = 0,$$

$$c\mathbb{1} = \mathbb{1}.$$

The space of the affine $\hat{gl}_N$ vertex algebra at level 1 is the induced $\hat{gl}_N$-module

$$V_{gl} = Ind_{\hat{gl}_N^0 \oplus \hat{gl}_N^+}^{\hat{gl}_N} \mathbb{C}\mathbb{1} \cong U(\hat{gl}_N^-) \otimes \mathbb{1}.$$