Abstract

Mighty oaks from little acorns grow

Sphere-packing, the Leech lattice and the Conway group

Rob Curtis

CIMPA Conference July 2015

Indispensable references

1. **H.S.M. Coxeter**,

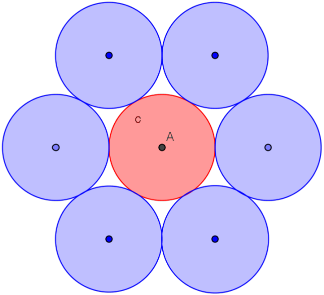
   Introduction to Geometry

   Wiley 1961.
2. **J.H. Conway and N.J.A. Sloane**,

   Sphere Packings, Lattices and Groups

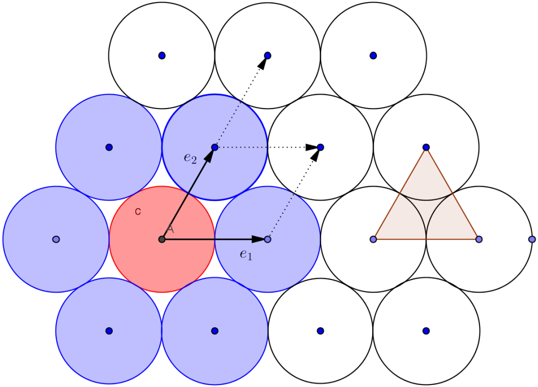
   Springer-Verlag 1988.

The Kissing Number



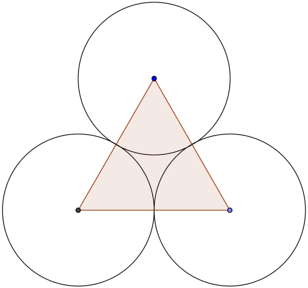
The pink circle is touched by 6 non-overlapping blue circles: The Kissing Number in $\mathbb{R}^2$ is 6.

A Lattice Packing



The centres of the circles lie on the lattice $\Lambda = \{me_1 + ne_2 \mid m, n \in \mathbb{Z}\}$. The plane is covered by triangles congruent to the one indicated.

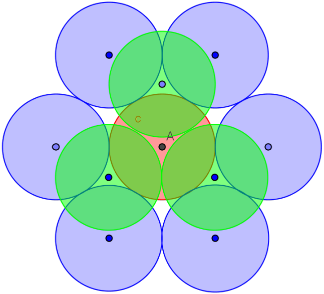
The density of a lattice packing



The density of the hexagonal lattice in $\mathbb{R}^2$ is

$$\frac{\pi/2}{\frac{1}{2}.2\sqrt{3}} = \frac{\pi}{2\sqrt{3}} \sim .9069$$

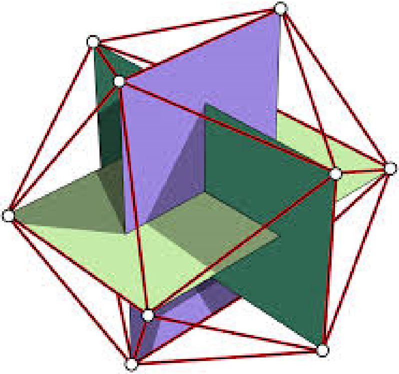
The Kissing Number in $\mathbb{R}^3$



Visibly we can have 12 unit spheres touching a given unit sphere without overlapping one another. So the Kissing number in $\mathbb{R}^3$ is at least 12.

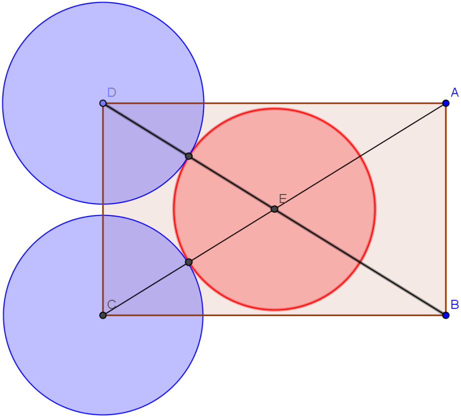
Isaac Newton 1643-1727 and David Gregory 1659-1708

Kissing number in $\mathbb{R}^3$.



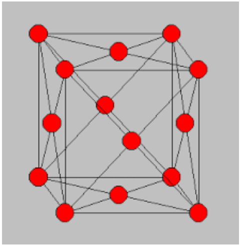
The vertices of 3 golden rectangles mutually perpendicular to one another lie at the 12 vertices of a regular icosahedron. cf. Coxeter's *Geometry* page 162 following Fra Luca Pacioli 1445-1509 *De divina proportione*.

Golden Rectangle



At each vertex of the icosahedron place a sphere with centre that vertex and radius r one half the distance of the vertex from O, the centre of the icosahedron. These spheres all touch a sphere of radius r centre O but do not touch one another.

Highest density of a lattice packing in $\mathbb{R}^3$



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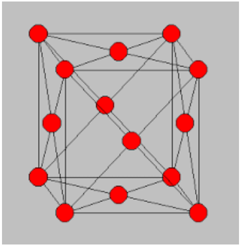

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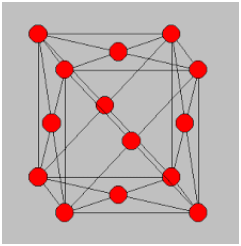

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- This has density $\pi/\sqrt{18} \sim \cdot 74048$. Rogers: "many mathematicians believe, and all physicists know" that this is best possible. [C-S]

Calculation of the density Δ

- A generator matrix M and Gramm matrix $A = MM^t$ for D_3 are given by

$$M = \begin{pmatrix} -1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \text{ and } A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

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- $\mathrm{V}_n(R) = \dfrac{2\pi R^2}{n}\mathrm{V}_{n-2}(R) = \dfrac{\pi^{n/2}}{(n/2)!}R^n.$

Higher dimensions: sphere-packing in $\mathbb{R}^n$

- The distance between 2 points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ is defined to be

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- The general question is too hard, so usually restrict to lattice packings.

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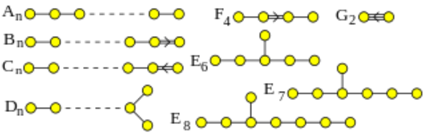
- So 24 spheres of radius $\sqrt{2}/2$ with centres at these points will all touch a central sphere of the same radius and will not overlap.
- Oleg Musin (2003) proved that this is best possible, so $\tau_4 = 24$. The problem is equivalent to asking how many points can be placed on S_{n-1} so that the *angular separation* between any two of them is at least $\pi/3$.

Coxeter-Dynkin diagrams

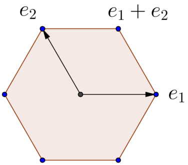


Crystallographic finite reflection groups. A reflection in the hyperplane orthogonal to a root r, given by

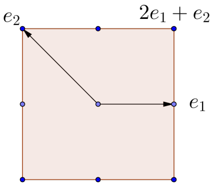
$$\theta_r : x \mapsto x - 2\frac{x.r}{r.r}r,$$

preserves the lattice Λ.

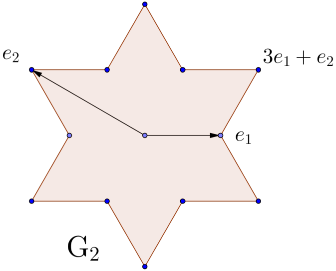
The 2-dimensional chrystallographic lattices



A_2



B_2



G_2

Best known kissing numbers and packings C&S 1988

Table 1.1. Records for packings, kissing numbers, coverings and quantizers. (Box: optimal. To left of double line: known to be optimal among lattices.) For $n \leqslant 8$ the entry in the first row is $\cong \Lambda_n$.

DIMENSION	1	2	3	4	5	6	7	8	12	16	24
DENSEST PACKING	Z	A_2	A_3	D_4	D_5	E_6	E_7	E_8	K_{12}	Λ_{16}	Λ_{24}
HIGHEST KISSING NUMBER	Z 2	A_2 6	A_3 12	D_4 24	D_5 40	E_6 72	E_7 126	E_8 240	K_{12} 756	Λ_{16} 4320	Λ_{24} 196560
THINNEST COVERING	Z	A_2	A_3^*	A_4^*	A_5^*	A_6^*	A_7^*	A_8^*	A_{12}^*	A_{16}^*	Λ_{24}
BEST QUANTIZER	Z	A_2	A_3^*	D_4	D_5^*	E_6^*	E_7^*	E_8	K_{12}	Λ_{16}	Λ_{24}

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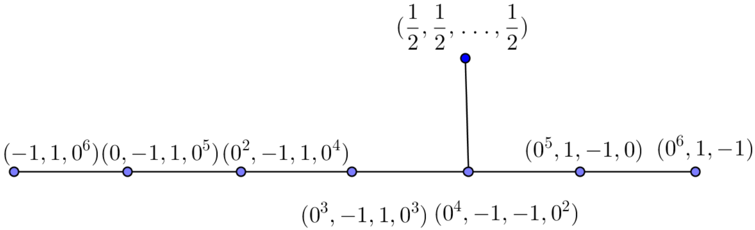
- An integral lattice Λ such that $x.x \in 2\mathbb{Z}$ for all $x \in \Lambda$ is said to be even. Even unimodular lattices exist if, and only if, dimension $n = 8k$. One for $n = 8$; two for $n = 16$; twenty-four for $n = 24$, the Niemeier lattices

The E_8 lattice



The shortest vectors in Λ and the Weyl group of E_8

- $$\Lambda = \left\{ (x_1, x_2, \ldots, x_8) \mid \sum x_i \in 2\mathbb{Z} \text{ and } \left\{ \begin{array}{l} \textbf{either } x_i \in \mathbb{Z} \text{ for all } i \\ \textbf{or } x_i \in \mathbb{Z} + \frac{1}{2} \text{ for all } i \end{array} \right. \right\}$$

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- 240 spheres of radius $\frac{1}{\sqrt{2}}$ centred at these lattice points all touch a sphere centre O of the same radius and do not overlap. $\tau_8 = 240$. [Odlyzko and Sloane 1979, Chapter 13 in CS]

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- The supports of these codewords are known as $\mathcal{C}$-sets.

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  (iv) 759 codewords of weight 16, the 16-ads complements of octads
  (v) a codeword of weight 24, the all ones vector.

$$1 + 759 + 2576 + 759 + 1 = 2^{12}.$$

- The supports of these codewords are known as $\mathcal{C}$-sets.
- The group of permutations of the 24 coordinates preserving $\mathcal{C}$ is the quintuply transitive Mathieu group M_{24} of order 244, 823, 040. Every subset of 5 points lies in precisely one octad.

John Leech 1926-92, Skipper of the *Waverley*

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  With this scaling every lattice vector has norm $\sum x_i^2 = 16n$; such a vector is said to be of type Λ_n.

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- So we can place 196560 non-overlapping spheres with radius $\frac{1}{2}.\sqrt{16.2} = 2\sqrt{2}$ and centres at these lattice points and they will all touch a sphere of the same radius centred on the origin. It turns out that this is best possible and the kissing number $\tau_{24} = 196560$. [Odlyzko and Sloane 1979, Chapter 13 in CS.]

John McKay 1939 - : " a snapper up of unconsidered trifles "

John Horton Conway 1937 - for whom Mathematics is a Game, and Games are Mathematics.

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- Wish to go straight from H to G, obtaining Λ as a by-product.

Symmetric presentation of $\cdot\mathrm{O}$

- Suppose that there is a group G generated by a set of $\binom{24}{4}$ involutions, corresponding to tetrads of the 24 points on which M_{24} acts, and which are permuted within G by inner automorphisms corresponding to M_{24}. So have a homomorphism

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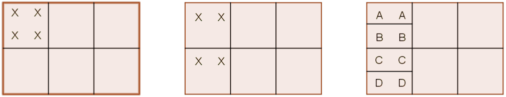

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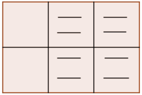
$\nu =$ 

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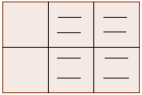
$\nu =$ 

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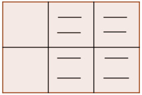
$\nu =$ 

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- So factor out this relation to obtain

$$G = \frac{2^{\star\binom{24}{4}} : \mathrm{M}_{24}}{\nu = t_{AB} t_{AC} t_{AD}} \cong \cdot\mathrm{O}, \text{ The Conway group.}$$

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- Conway: the group $\cdot\mathrm{O}$ is simply M_{24} writ large