The Thompson chain of perfect groups

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- So any group generated by three involutions is a homomorphic image of E.
- Clearly the group $m^{\star n}$ possesses monomial automorphisms which permute the symmetric generators and raise them to powers co-prime to m. The group M of all monomial automorphisms of $m^{\star n}$ thus has order

$$|M| = n!\ \phi(m)^n, \text{ where } \phi \text{ is the Euler totient function.}$$

Motivation

- If $H \leq M$ then we can thus form a semi-direct product of shape $m^{\star n} : H$, a progenitor. The elements $\{t_1, t_2, \ldots, t_n \mid t_i^m = 1\}$ are the symmetric generators.

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- The group $3{}^{\cdot}\mathrm{A}_7$ has a subgroup of index 15 of shape $3 \times \mathrm{L}_2(7)$ and thus has a 15-dimensional monomial representation over any field with cube roots of unity, such as $\mathbb{Z}_7$.

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- This representation enables us to define a progenitor of shape $\mathsf{P} = 7^{\star 15} : 3{}^{\cdot}\mathrm{A}_7$ in which the central 3-element squares each of the symmetric generators by conjugation.
- Now A_7 acts on 15 letters in two distinct ways and it is useful to "double up" to a progenitor of form

$$P = 7^{\star(15+15)} : 3{}^{\cdot}\mathrm{S}_7$$

in which the "central" 3 squares one set of 15 symmetric generators while fourth powering the other 15.

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- $$\frac{7^{\star(15+15)} : 3{\cdot}\mathrm{S}_7}{(dt_1)^3} \cong \mathrm{He}, \text{ the Held sporadic simple group.}$$

An alternative progenitor

- A group of shape $3 \times \mathrm{S}_n$ has a subgroup to index $\binom{n}{2}$ of shape $6 \times \mathrm{S}_{n-2}$ and so $3 \times \mathrm{S}_n$ has an $\binom{n}{2}$ monomial representation over any field containing 6th roots of unity. Again $\mathbb{Z}_7$ is a candidate.

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- So we may form the progenitor

$$P_n = 7^{\star\binom{n}{2}} : (3 \times \mathrm{S}_n),$$

where a central element z of order 3 squares each of the symmetric generators, and ask what (almost) simple images such a group might have.

The complete graph K_n

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  (i) an automorphism which squares each of them;
  (ii) an automorphism which cycles them, and
  (iii) an automorphism which inverts one them, while interchanging another with the inverse of the third.
- Among such groups are $\mathrm{L}_2(8):3, \quad \mathrm{S}_7, \quad \mathrm{U}_3(3):2$.

The triangle progenitor P_3.

A free product of three copies of C_7 on which certain automorphisms have been imposed.

$$7^{\star 3} : \qquad (3 \times \mathrm{S}_3)$$

$$\langle t_{12}, t_{23}, t_{31} \rangle \quad \underset{z}{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}} \underset{(1\ 2\ 3)}{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}} \underset{(1\ 2)}{\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}}.$$

The unitary group $U_3(3)$.

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- We let

$$A := \begin{pmatrix} 1+i & i & i \\ i & 1+i & i \\ i & i & 1+i \end{pmatrix};\ Z := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

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- Matrix A, which has order 3, is visibly unitary and symmetric and so conjugation by α inverts it; moreover Z, which corresponds to a rotation of the three coordinates, commutes with A. Thus

$$\langle Z, A, \alpha \rangle \cong 3 \times \mathrm{S}_3.$$

Symmetric generators in $\mathrm{U}_3(3)$.

As symmetric generators we take $ts := [t_{12}, t_{23}, t_{31}] =$

$$\left[\begin{pmatrix} i & 1 & -1-i \\ 1 & 1-i & i \\ -1-i & i & -i \end{pmatrix}, \begin{pmatrix} 1 & -1 & -1-i \\ -1+i & -1+i & 0 \\ 1 & -1 & 1+i \end{pmatrix}, \begin{pmatrix} 1 & -1+i & 1 \\ -1 & -1+i & -1 \\ -1-i & 0 & 1+i \end{pmatrix}\right]$$

and find that $t_{12}t_{23}^4t_{12}t_{23}^3t_{12}^3 = A^2Z$.

- If we let $x = Z\alpha, y = A, t = t_{12}$ then we obtain the presentation

$$\langle x, y, t \mid x^6 = y^3 = y^x y = t^7 = t^x t^2 = t(t^y)^4 t(t^y)^3 t^3 x^2 y = 1\rangle \cong \mathrm{U}_3(3) : 2.$$

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12096
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- We define $t_{ij}^d = s_{ij}$, then $\langle t_{ij}, s_{ij} \rangle \cong \mathrm{L}_2(7)$

Extending to higher values of n.

- As generators for $3 \times \mathrm{S}_n$ we take $x = (1\ 2\ \ldots\ n)$ and $y = (1\ 2)z$ when we see that

$$x^n = y^6 = [x, y^2] = (xy^3)^{n-1} = [x, y]^3 = [x^2, y]^2 = 1$$

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4 72
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6 2160
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- In order to obtain an expression for our additional relation $R = 1$ we note that $[x,y] = (1\ 2\ 3)$, so may write

$$R = t(t^x)^4 t(t^x)^3 t^3 y^2 [x,y].$$

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- We then define

$$K_n = \frac{7^{\star\binom{n}{2}} : (3 \times \mathrm{S}_n)}{R = 1}.$$

Coset Enumeration along the chain

- Note that $xy = (2\ 3\ \ \ldots n)z$ and $y^x = (2\ 3)z$ generate $3 \times \mathrm{S}_{n-1}$ and so $\langle xy, y^x, t^x \rangle$ generates (an image of) K_{n-1}. We perform a coset enumeration of K_n over this subgroup.

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Those familiar with the groups involved will recognise that

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K7: 3091200 = 2 x 1545600 is the index of $3\cdot\mathrm{Suz}$ in Co_1, the largest Conway simple group.

Higher values of n

- The groups defined by our family of presentations are

$$\mathrm{U}_3(3):2,\quad \mathrm{HJ}:2,\quad \mathrm{G}_2(4):2,\quad 3^{\cdot}\mathrm{Suz}:2\ \text{ and }\ \mathrm{Co}_1\times 2.$$

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- In order to proceed we must factor out the central element in K_7 to obtain Co_1. This leads to K_8 and subsequent groups collapsing to the trivial group.
- However, if we restrict our imposed automorphisms to $3\times \mathrm{A}_n$, then $KA_9 \cong \mathrm{Co}_1$, in which a 3-cycle such as (1 2 3) lies in the centre of the copy of $3^{\cdot}\mathrm{Suz}$ generated by the complete 6-graph on the remaining 6 points.

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(4) Each 4-graph corresponds to a copy of HJ;

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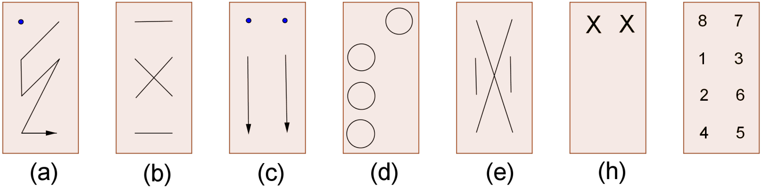
- For convenience we label the 24 points as shown

8	7	16	15	24	23
1	3	9	11	17	19
2	6	10	14	18	22
4	5	12	13	20	21

The centralizer of the MOG group



Label	Action in $L_2(7)$	Action in A_9
(a)	(1 2 3 4 5 6 7)	(1 2 3 4 5 6 7)
(b)	(8 7)(1 6)(2 3)(4 5)	(8 7)(1 6)(2 3)(4 5)
(c)	(1 2 4)(3 6 5)	(1 2 4)(3 6 5)
(d)		(8 7)(1 3)(2 6)(4 5)
(e)	(8 5)(7 4)(1 2)(3 6)	(8 5)(7 4)(1 2)(3 6)
(h)		(1 4 2)(3 6 5)
(j)		(8 9)(1 6)(2 5)(3 4)

Completing A_8 to A_9

We seek matrix J representing $j = (8\ 9)(1\ 6)(2\ 5)(3\ 4)$ which inverts (a) and commutes with (c), represented by matrices A and C respectively.

- $$B = \langle \begin{bmatrix} 0 & I_8 & 0 \\ 0 & 0 & I_8 \\ I_8 & 0 & 0 \end{bmatrix}, \begin{bmatrix} I_8 & 0 & 0 \\ 0 & 0 & I_8 \\ 0 & I_8 & 0 \end{bmatrix} \rangle \cong \mathrm{S}_3$$

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- Any matrix commuting with B has shape

$$\begin{bmatrix} X & Y & Y \\ Y & X & Y \\ Y & Y & X \end{bmatrix}$$

which enables us to write down the precise form of J.

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$$X := \begin{pmatrix} 1 & 4 & 1 & 4 & 4 & 1 & 1 & 1 \\ 4 & 1 & 4 & 4 & 1 & 1 & 1 & 1 \\ 1 & 4 & 4 & 1 & 1 & 1 & 4 & 1 \\ 4 & 4 & 1 & 1 & 1 & 4 & 1 & 1 \\ 4 & 1 & 1 & 1 & 4 & 1 & 4 & 1 \\ 1 & 1 & 1 & 4 & 1 & 4 & 4 & 1 \\ 1 & 1 & 4 & 1 & 4 & 4 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 5 \end{pmatrix} ; \ Y := \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 4 & 1 & 1 \\ 1 & 1 & 1 & 1 & 4 & 1 & 1 & 1 \\ 1 & 1 & 1 & 4 & 1 & 1 & 1 & 1 \\ 1 & 1 & 4 & 1 & 1 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 4 \end{pmatrix}$$

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- The matrix U satisfies $U^2 = 4U - 4I;\ UU^t = 4I = U + U^t$, so verification that T_{89} has required properties is straightforward.