§1 Jansian topologizing filters

<u>Theorem</u>. If $\mathcal{F} \in \mathrm{Fil}\,R_R$, then $I = \bigcap \mathcal{F}$ is a two-sided ideal of R.

<u>Proof</u>. Put $I = \bigcap \mathcal{F}$. For each $r \in R$,

$$r^{-1}I = r^{-1}(\bigcap \mathcal{F}) = \bigcap_{K \in \mathcal{F}} r^{-1}K.$$ Since $r^{-1}K \in \mathcal{F}\ \forall K \in \mathcal{F}$,

$$\bigcap_{K\in\mathcal{F}} r^{-1}K \supseteq \bigcap \mathcal{F} = I.$$ Thus $r^{-1}I \supseteq I\ \forall r \in R$.

It follows that I is two-sided. $\square$

<u>Theorem</u>. The following statements are equivalent for $\mathcal{F} \in \mathrm{Fil}\,R_R$:

(1) $\mathcal{F}$ is closed under arbitrary intersections;

(2) $\bigcap \mathcal{F} \in \mathcal{F}$;

(3) $\mathcal{F} = \eta(I)$ for some (two-sided) ideal I of R;

(4) $\mathcal{F}$ has a smallest member, i.e., $\mathcal{F}$ is principal.

$\square$

<u>Definition</u>. We call $\mathcal{F} \in \mathrm{Fil}\,R_R$ <u>Jansian</u> if it satisfies the above equivalent conditions.

Observe that the Jansian members of $\mathrm{Fil}\,R_R$ are in one-to-one correspondence with the members of $\mathrm{Id}\,R$. This correspondence can be stated more precisely thus.

Theorem. The map $\eta : \operatorname{Id} R \longrightarrow \operatorname{Fil} R_R$

$$I \longmapsto \eta(I)$$

is one-to-one and order reversing; it maps arbitrary joins in $\operatorname{Id} R$ to meets in $\operatorname{Fil} R_R$, and finite meets in $\operatorname{Id} R$ to joins in $\operatorname{Fil} R_R$. □

Corollary. $\operatorname{Id} R$ embeds as a sublattice in $[\operatorname{Fil} R_R]^{du}$, the order dual of $\operatorname{Fil} R_R$. □

The above theorem tells us that $\operatorname{Fil} R_R$ encodes at least as much information about the ring R as does $\operatorname{Id} R$.

Question When is the map $\eta : \operatorname{Id} R \longrightarrow \operatorname{Fil} R_R$ onto?

Theorem [Beachy & Blair, 1978] The following statements are equivalent for a ring R:

(1) Every $F \in \operatorname{Fil} R_R$ is Jansian, that is, has the form $F = \eta(I)$ for some $I \in \operatorname{Id} R$;

(2) R is right artinian, i.e., R satisfies the DCC on right ideals.

Proof. (2) $\Rightarrow$ (1) (only). An immediate consequence of the fact that in a right artinian ring, every nonempty family of right ideals has a smallest member. □

Imposing triviality on $\mathrm{Fil}\, R_R$ amounts to a stronger condition than imposing triviality on $\mathrm{Id}\, R$, since, as we have seen, the latter structure is in general smaller than the former. This distinction is bourne out by a juxtaposition of the following:

- [Theorem in Lecture 3] $\underline{\mathrm{Fil}\, R_R \text{ is trivial}}$, i.e., $\mathrm{Fil}\, R_R = \{\{R\}, \eta(0)\} \iff R$ is $\underline{\text{simple artinian}}$, so $R \cong \mathbb{M}_n(D)$, D a division ring.

- $\underline{\mathrm{Id}\, R \text{ is trivial}}$, i.e., $\mathrm{Id}\, R = \{0, R\} \iff R$ is $\underline{\text{simple}}$.

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Lecture 5

Structure on the set of topologizing filters – $\mathrm{Fil}\,R_R$ as a lattice ordered monoid

§1 A binary operation on $\mathrm{Fil}\,R_R$

We shall add to the structural richness of $\mathrm{Fil}\,R_R$ by defining a binary operation on $\mathrm{Fil}\,R_R$.

$\forall F, G \in \mathrm{Fil}\,R_R$

$$F : G \overset{\text{def}}{=} \{K \leq R_R : \exists H \in F \text{ s.t. } H \supseteq K \ \& \ h^{-1}K \in G \ \forall h \in H\}.$$

The above definition can be recast in a form that makes it easier to grasp. We first need to introduce the notion of torsion with respect to a topologizing filter.

Definition. Let $F \in \mathrm{Fil}\,R_R$. We say that a right R-module M is *F-torsion* if
$\forall x \in M, \exists K \in F$ s.t. $xK = 0$.

Note. The family $F = \{n\mathbb{Z} \leq \mathbb{Z} : n \in \mathbb{N}\}$ of all nonzero ideals of $\mathbb{Z}$ is easily seen to be a topologizing filter on $\mathbb{Z}$.
A $\mathbb{Z}$-module (i.e., abelian group) G is F-torsion
iff $\forall g \in G, \exists n \in \mathbb{N}$ s.t. $gn = 0$
iff G is torsion in the classical sense.

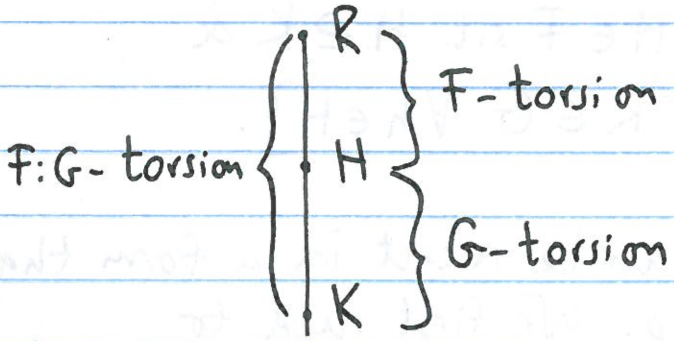
Theorem. Let $F, G \in \mathrm{Fil}\,R_R$. Then:

(1) $K \in F \iff R/K$ is F-torsion.

(2) $K \in F:G \iff$ for some $H \leq R_R$ the short exact sequence

$$0 \longrightarrow H/K \longrightarrow R/K \longrightarrow R/H \longrightarrow 0$$

is such that H/K is G-torsion & R/H is F-torsion.



Thus $F:G$ corresponds with a type of 'extension'.

Note. If $F, G \in \mathrm{Fil}\,R_R$, then $F:G \supseteq F \,\&\, G$.

§2 Lattice ordered monoids

Definition A *lattice ordered monoid* is a structure $\langle L; \leq, \cdot; e_L \rangle$ where:

(L1) $\langle L; \leq \rangle$ is a lattice;
(L2) $\langle L; \cdot; e_L \rangle$ is a monoid with identity e_L;
(L3) $\forall a, b, c \in L$

$$a \cdot (b \vee c) = (a \cdot b) \vee (a \cdot c)$$
$$\& \quad (b \vee c) \cdot a = (b \cdot a) \vee (c \cdot a).$$

[L3 means that '$\cdot$' respects the order relation, that is to say, $x \le y \Rightarrow x \cdot a \le y \cdot a$ & $a \cdot x \le a \cdot y$ $\forall a \in L$.]

If L has a top element 1_L that coincides with the monoid identity e_L, we call L <u>integral</u>.

<u>Definition</u>. Let L be a lattice ordered monoid. We say L is <u>left residuated</u> [resp. <u>right residuated</u>] if $\forall a, b \in L$, $\exists$ largest $x \in L$ s.t. $x \cdot b \le a$ [resp. $b \cdot x \le a$]; we call such an x the <u>left residual of a by b</u> [resp. <u>right residual of a by b</u>] and denote it ab^{-1} [resp. $b^{-1}a$].

[If pressed for time omit following theorem.]

<u>Theorem</u> The following statements are equivalent for a complete lattice ordered monoid L:

(1) L is left residuated;

(2) $(\bigvee X) \cdot a = \bigvee_{x \in X} (x \cdot a)$ $\forall a \in L, \forall X \subseteq L$.

$\square$

For Ring theorists, the following is the prototype of lattice ordered monoid.

<u>Example</u>. Let R be any ring (with identity). Then

$$\langle \mathrm{Id}\, R; \cap, +, \cdot, R \rangle$$

is a lattice ordered, left & right residuated, integral monoid.

(i) The meet is intersection.
(ii) The join is ideal addition $+$.
(iii) The binary operation '$\cdot$' is ideal multiplication: if $I, J \in \mathrm{Id}R$, then

$$I \cdot J = \left\{ \sum_{i=1}^{n} a_i b_i : a_i \in I \ \& \ b_i \in J, \ 1 \le i \le n \right\}.$$

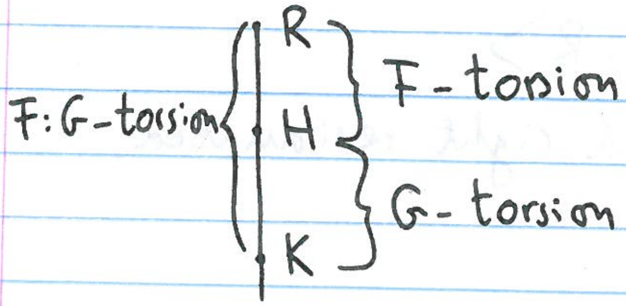
(iv) The ideal R is the identity w.r.t '$\cdot$' since $\forall I \in \mathrm{Id}R$, $I \cdot R = R \cdot I = I$.

(v) If $I, J \in \mathrm{Id}R$, then
$IJ^{-1} = \{t \in R : tJ \subseteq I\}$ = left residual of I by J
& $J^{-1}I = \{t \in R : Jt \subseteq I\}$ = right residual of I by J.

<u>Theorem</u>. The order dual of
$\langle \mathrm{Fil}\, R_R ; \cap, \vee, :, \{R\} \rangle$ is a lattice ordered, <u>left residuated</u>, integral monoid.

[Omit proof if pressed for time.]

<u>Proof</u>. (Only that $\{R\}$ is the monoid identity).
Note that if $F = \{R\}$, then a right R-module M is F-torsion $\Leftrightarrow \forall x \in M$, $xR = 0 \Leftrightarrow M = 0$.



Now $K \in \mathcal{F} : \mathcal{G}$

$\iff \exists H \leq R_R$ s.t. R/H is $\mathcal{F}$-torsion & H/K is $\mathcal{G}$-torsion.

$\iff R/H = 0$ & H/K is $\mathcal{G}$-torsion

$\iff R/K$ is $\mathcal{G}$-torsion

$\iff K \in \mathcal{G}$.

Thus $\mathcal{F} : \mathcal{G} \subseteq \mathcal{G}$. Since the reverse containment always holds, we must have $\mathcal{F} : \mathcal{G} = \mathcal{G}$.

Proof that $\mathcal{G} : \mathcal{F} = \mathcal{G}$ is similar. $\square$

<u>Note</u>. In general $[\mathrm{Fil}\, R_R]^{du}$ is <u>not</u> right residuated.

We now return to, and extend, a theorem from Lecture 4.

<u>Theorem</u>. The map $\eta : \mathrm{Id} R \longrightarrow [\mathrm{Fil}\, R_R]^{du}$ is :
$$I \longmapsto \eta(I)$$

(i) one-to-one
(ii) join preserving
(iii) finite meet preserving ; and

$\}$ stated in earlier version of theorem

(iv) multiplicative, that is, $\forall I, J \in \mathrm{Id} R$,
$$\eta(I \cdot J) = \eta(I) : \eta(J).$$ $\square$

Lecture 6 — Ring properties encoded in topologizing filters - some recent results

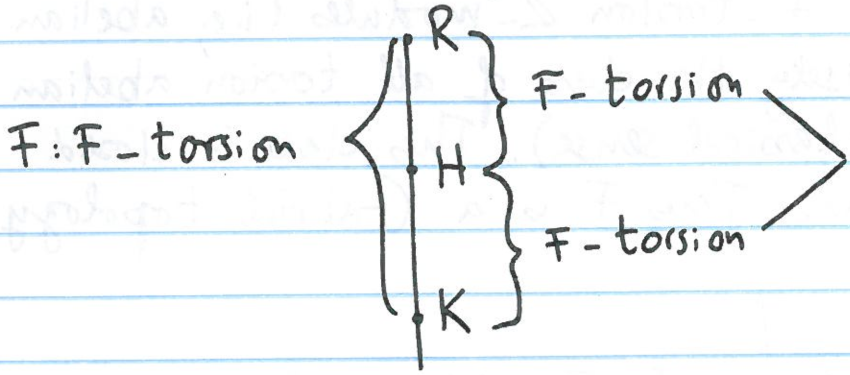
§1 Gabriel topologies

<u>Theorem</u>. The following statements are equivalent for $\mathcal{F} \in \mathrm{Fil}\,R_R$:

(1) $\mathcal{F}$ is idempotent, i.e. $\mathcal{F} : \mathcal{F} = \mathcal{F}$;

(2) The class of all $\mathcal{F}$-torsion modules is closed under extensions.

<u>Proof</u> ((2) $\Rightarrow$ (1) only). If $K \in \mathcal{F} : \mathcal{F}$ we have,



(2) implies R/K is $\mathcal{F}$-torsion, i.e. $K \in \mathcal{F}$

The above shows $\mathcal{F} : \mathcal{F} \subseteq \mathcal{F}$, whence $\mathcal{F} : \mathcal{F} = \mathcal{F}$. $\square$

<u>Definition</u>. We call $\mathcal{F} \in \mathrm{Fil}\,R_R$ a <u>right Gabriel topology</u> on R if $\mathcal{F}$ satisfies the above equivalent conditions.

Gabriel topologies on a ring R are important because it is possible to define an abstract notion of localization w.r.t any right Gabriel topology over R. Indeed, topologizing filters were invented with this

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application in mind.

<u>Notation</u>. $\mathrm{Gab}\,R_R$ = set of all right Gabriel topologies on R.

<u>Note</u>. $\mathrm{Gab}\,R_R$ is a <u>frame</u> (a lattice in which meets distribute over arbitrary joins). It is not, however, a sublattice of $\mathrm{Fil}\,R_R$; meets are intersections, but joins in $\mathrm{Gab}\,R_R$ differ from those in $\mathrm{Fil}\,R_R$.

<u>Example</u>. Define $\mathcal{F} = \{n\mathbb{Z} \leq \mathbb{Z} : n \in \mathbb{N}\}$ = set of all nonzero ideals of $\mathbb{Z}$.
The class of all $\mathcal{F}$-torsion $\mathbb{Z}$-modules (i.e. abelian groups) is precisely the class of all torsion abelian groups (in the classical sense). This class is closed under extensions. Thus $\mathcal{F}$ is a Gabriel topology on $\mathbb{Z}$.

Given the importance of Gabriel topologies it is natural to ask:

<u>Question</u> [Teply, Viola-Prioli, vdB]
For what rings R is $\mathrm{Fil}\,R_R = \mathrm{Gab}\,R_R$?

The commutative case can be easily disposed of:

<u>Theorem</u> [J. Viola-Prioli, 1975] The following statements are equivalent for a commutative ring R:

(1) $\mathrm{Fil}\,R_R = \mathrm{Gab}\,R_R$;
(2) R is a finite product of fields. $\square$

No satisfactory characterization of (possibly noncommutative) rings R for which $\mathrm{Fil}\,R_R = \mathrm{Gab}\,R_R$ exists. Some properties of such rings are listed below.

<u>Theorem</u>

(1) If $\mathrm{Fil}\,R_R = \mathrm{Gab}\,R_R$, then $\mathrm{Id}\,R$ is distributive;
(2) If R is a right noetherian, right PCI ring (right PCI means 'Proper Cylic Injective': if C is cyclic and $C \not\cong R_R$, then C is injective).
(3) $\exists$ non-noetherian rings R for which $\mathrm{Fil}\,R_R = \mathrm{Gab}\,R_R$ [vdB 1999]. (Example is a non-noetherian right chain domain with only three ideals.) $\square$

§2 <u>Prime and strongly prime rings</u>

<u>Theorem</u>. The following statements are equivalent for a ring R:
(1) $\forall I, J \in \mathrm{Id}\,R$, $I \cdot J = 0 \Rightarrow I = 0$ or $J = 0$;
(2) $\forall a, b \in R$, $aRb = 0 \Rightarrow a = 0$ or $b = 0$.

A ring R satisfying the above equivalent conditions is called <u>prime</u>.

A ring R is said to be <u>right strongly prime</u> if $\forall$ nonzero $a \in R$ $\exists$ <u>finite</u> subset $X \subseteq R$ s.t $\forall b \in R$ $aXb = 0 \Rightarrow b = 0$.

The notion of strong primeness arose naturally in the study of primitivity conditions in certain classes of group rings.

The following result is implicit in Handelman & Lawrence's pioneering work on strongly prime rings in 1975.

<u>Theorem</u>. The following statements are equivalent for a ring R:

(1) $\forall \mathcal{F}, \mathcal{G} \in \mathrm{Fil}\, R_R$, $\mathcal{F} : \mathcal{G} = \eta(0) \Rightarrow \mathcal{F} = \eta(0)$ or $\mathcal{G} = \eta(0)$;
(2) $\mathrm{Fil}\, R_R$ contains a unique coatom & this coatom is idempotent;
(3) R is right strongly prime.

□

The above theorem tells us that right strong primeness is characterizable by means of a statement in the language of the lattice ordered monoid of right topologizing filters. This has the consequence that right strong primeness is a Morita invariant property. (This is a form of equivalence between rings that is much weaker than isomorphism, for example, R and $\mathbb{M}_n(R)$ are Morita equivalent for all rings R and all $n \in \mathbb{N}$.)

The following example is similar to the example in §1.

<u>Example</u>. Let R be a commutative domain (e.g. $R = \mathbb{Z}$). Since R contains no zero-divisors it is right strongly prime, so by the previous theorem, $\mathrm{Fil}\, R_R$ must possess a unique coatom $\mathcal{F}$ which is

idempotent. The Gabriel topology $\mathcal{F} = \{I \leq R : I \neq 0\}$ = set of all nonzero ideals of R.

§3 Commutativity conditions on $\mathrm{Fil}\,R_R$

Recall that (the order dual of) $\mathrm{Fil}\,R_R$ is:

- Left residuated, meaning $\forall F, G \in \mathrm{Fil}\,R_R$, $\exists$ smallest H s.t. $H : F \supseteq G$.
- in general not right residuated.

Questions [Arega Chere, vdB]

- When is $\mathrm{Fil}\,R_R$ left and right residuated?
- [Stronger] When is $\mathrm{Fil}\,R_R$ commutative (meaning, the binary operation $:$ is commutative)?

Note. If R is a commutative ring, then $\mathrm{Fil}\,R_R$ need not be commutative.

Theorem [Arega Chere, vdB] The following statements are equivalent for a commutative ring R:

(1) $\mathrm{Fil}\,R_R$ is left and right residuated;
(2) $\mathrm{Fil}\,R_R$ is commutative. $\square$

Theorem [vdB, 1999] Let R be a commutative ring. If R is noetherian, then $\mathrm{Fil}\,R_R$ is commutative, but not conversely. $\square$

Notwithstanding the above, commutative rings R for which $\mathrm{Fil}\,R_R$ is commutative enjoy many 'noetherian-like' properties:

<u>Theorem</u>. [vdB, 1999] Let R be a commutative ring for which $\mathrm{Fil}\,R_R$ is commutative. Then:

(1) R satisfies the ACC on principal ideals;

(2) The prime radical of R is nilpotent;

(3) Every indecomposable injective right R-module has the form $E(R/P)$ for some prime ideal P of R. $\square$

<u>Question</u>. If R is an arbitrary right noetherian ring, is $\mathrm{Fil}\,R_R$ left and right residuated?

<u>Partial answer</u> [Arega Chere, vdB]

Yes if R is <u>right fully bounded</u> noetherian (these are right noetherian rings that satisfy a 'weak-commutativity' condition).

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