Lecture 1 Topological rings, the absolute value and field valuations

References: Bourbaki, Zariski & Samuel, Google!

§1 Topological Rings

Definition. A topology T on a ring R is called a *ring topology* if $+ : R^2 \longrightarrow R$ and $\cdot : R^2 \longrightarrow R$ are continuous w.r.t T.
We call (R, T) a *topological ring*, but write just R if T is understood.

$\mathbb{R}$ together with the usual topology is an example of a topological ring. In this case the topology is induced by the usual metric d on $\mathbb{R}$ where

$$d(x,y) \overset{\text{def}}{=} |x-y| \quad \forall x,y \in \mathbb{R}.$$

The *absolute value function* $|\cdot| : \mathbb{R} \longrightarrow \mathbb{R}^+ \cup \{0\}$ satisfies:

(A1) $|x| = 0 \Leftrightarrow x = 0$.

(A2) $|xy| = |x||y| \quad \forall x,y \in \mathbb{R}$.

(A3) $|x+y| \leq |x| + |y| \quad \forall x,y \in \mathbb{R}$ [Δ-inequality]

Theorem. If R is a topological ring, then:

(1) $\delta_a : R \longrightarrow R$ is a homeomorphism for each $a \in R$.
$[\delta_a(x) \overset{\text{def}}{=} a + x \quad \forall x \in R]$;

(2) $\lambda_a : R \longrightarrow R$ and $\rho_a : R \longrightarrow R$ are continuous for each $a \in R$. [$\lambda_a(x) \stackrel{\text{def}}{=} ax$ & $\rho_a(x) \stackrel{\text{def}}{=} xa$ $\forall x \in R$.] $\square$

[Converse: if $+ : R^2 \longrightarrow R$ is continuous & λ_a, ρ_a are continuous $\forall a \in R$, then $\cdot : R^2 \longrightarrow R$ is continuous, thus making R a topological ring.]

Recall that if X is a topological space & $x \in X$, then a family $\mathcal{N}$ of subsets of X is called a <u>nbd base of opens</u> for x, if every member of $\mathcal{N}$ is open & every nbd of x contains some member of $\mathcal{N}$.

<u>Note</u> Let R be a topological ring. If $\mathcal{N}$ is a nbd base of opens for $0 \in R$, then $\delta_a[\mathcal{N}] = \{a+N : N \in \mathcal{N}\}$ is a nbd base of opens for a. Thus $\mathcal{B} = \bigcup_{a \in R} \delta_a[\mathcal{N}]$ is a base for the topology on R.

<u>Consequence</u>. In any topological ring R, the full topology on R can be recovered from any nbd base of opens for 0. (Indeed, it can be recovered from any nbd base of opens for any point in R.)

[Read following paragraph out, but don't write down if pressed for time.]

In a metric space, the family of all open balls centred at a particular point, constitutes a nbd base for that point. Thus if the topology on a topological ring R is induced by a metric, then, in light of the above, the family of open balls centred at 0 plays an especially important role.

§2 Valuations

The absolute value function is an example of a more general type of mapping called a valuation.

Definition. Let F be a field. A map $\nu : F \longrightarrow \mathbb{R}^+ \cup \{0\}$ is called a valuation on F if:

(V1) $\nu(x) = 0 \iff x = 0$;
(V2) $\nu(xy) = \nu(x)\nu(y) \quad \forall x, y \in F$;
(V3) $\nu(x+y) \leq \nu(x) + \nu(y) \quad \forall x, y \in F$ [Δ-inequality].

Stronger form of Δ-inequality:

(V3') $\nu(x+y) \leq \max\{\nu(x), \nu(y)\}$.

We call ν non-archimedean if it satisfies (V3'), otherwise, it is called archimedean.

Theorem. Let ν be a valuation on field F. Then:

(1) The map d_ν defined by $d_\nu(x,y) = \nu(x-y)$ $\forall x, y \in F$, is a metric on F;

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(2) The topology induced by the metric d_v is a ring topology on R. $\square$

<u>Example</u> The <u>p-adic valuation</u> [non-archimedean]

Let p be a fixed prime & $0<\alpha<1$.
The p-adic valuation $|\cdot|_p : \mathbb{Q} \longrightarrow \mathbb{R}^+ \cup \{0\}$ is defined as follows. Take $\frac{a}{b} \in \mathbb{Q}$.

- If $\frac{a}{b} = 0$, then $|\frac{a}{b}|_p \overset{\text{def}}{=} 0$.
- If $\frac{a}{b} \neq 0$, write $\frac{a}{b} = p^k \frac{a'}{b'}$, where $k \in \mathbb{Z}$ & $(a', p) = (b', p) = 1$. Then $|\frac{a}{b}|_p = |p^k \frac{a'}{b'}| \overset{\text{def}}{=} \alpha^k$.

E.g. $|p^2|_p = \alpha^2$; $|\frac{1}{p}|_p = \frac{1}{\alpha}$, etc.

Note that for each nonzero $q \in \mathbb{Q}$,
$$|q|_p \in \{\ldots \alpha^2, \alpha, 1, \tfrac{1}{\alpha}, \tfrac{1}{\alpha^2}, \ldots\}$$
Thus $|\cdot|_p$ is <u>discrete-valued</u>.

[If pressed for time, read following comment & theorem out, do not write down.]

In essence, the absolute value function & the p-adic valuation are the <u>only</u> valuations on $\mathbb{Q}$, in light of:

Theorem. Let ν be a non-trivial valuation on $\mathbb{Q}$.
(ν trivial means $\nu(x) = 1$ $\forall x \in \mathbb{Q}$, $x \neq 0$.)

(1) If ν is archimedean, then ν is equivalent to the absolute value $|\cdot|$ (meaning ν and $|\cdot|$ induce the same topology on $\mathbb{Q}$).

(2) If ν is non-archimedean, then ν is equivalent to the p-adic valuation $|\cdot|_p$ for some prime p. $\square$

§3 Restricting, extending and generalizing valuations

Note [completions]

(1) The completion of $\mathbb{Q}$ w.r.t the absolute value function, yields the reals $\mathbb{R}$. The absolute value function can be extended from $\mathbb{Q}$ to $\mathbb{R}$.

(2) The completion of $\mathbb{Q}$ w.r.t the p-adic valuation, yields the p-adic numbers $\mathbb{Q}_p$. The p-adic valuation can be extended from $\mathbb{Q}$ to $\mathbb{Q}_p$.

A valuation map extends uniquely from an integral domain (commutative ring without zero-divisors) to its field of quotients. This is a statement of the following theorem.

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Theorem Let ν be a valuation defined on an integral domain R. If F denotes the field of fractions of R, then ν is uniquely extendable to a valuation on F, defined thus

$$\nu\left(\frac{a}{b}\right) \stackrel{\text{def}}{=} \frac{\nu(a)}{\nu(b)} \quad \forall a, b \in R, \; b \neq 0.$$

$\square$

It follows from the above theorem that to define a valuation on $\mathbb{Q}$, for example, it suffices to define one on $\mathbb{Z}$.

The notion of a non-archimedean valuation on a field can be generalized by replacing the positive reals $\mathbb{R}^+$ with the positive cone of any multiplicative linearly ordered abelian group Γ. A valuation on a field F in this more general setting becomes a map $\nu : F \longrightarrow \Gamma^+ \cup \{0\}$ that satisfies axioms (V1), (V2) and (V3').

Such a <u>generalized valuation</u> also gives rise to a metric, defined in the obvious fashion, and this in turn induces a ring topology on F.

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Lecture 2 $\boxed{\text{Linear topologies and topologizing filters on rings}}$ $\boxed{L2_1}$

§1 Linear topologies

The only concrete examples of topological rings exhibited in Lecture 1 were fields (or subrings thereof) equiped with valuation maps. Ultimately, our interest will be in a certain type of ring topology that does not always have a valuation map at its root.

Here is our motivating example – it's one that we've already seen.

Example The p-adic valuation on $\mathbb{Z}$

p is a fixed prime & $0 < \alpha < 1$.
If $0 \neq a \in \mathbb{Z}$ write $a = p^k a'$ where $k \geq 0$ and $(a', p) = 1$. Then

$$|a|_p \overset{\text{def}}{=} \alpha^k.$$

The family of open balls $\mathcal{N} = \{B(0, \alpha^k) : k \geq 0\}$ is a nbd base of opens for 0, where

$$B(0,\alpha^k) \overset{\text{def}}{=} \{a \in \mathbb{Z} : |a|_p \leq \alpha^k\} = p^k\mathbb{Z} \ \forall k \geq 0.$$

Thus $\mathcal{N} = \{p^k\mathbb{Z} : k \geq 0\}$ constitutes a nbd base of opens for 0.

Observe that $\mathcal{N}$ comprises a family of ideals of $\mathbb{Z}$. We give a special name to a ring topology that admits such a nbd base for O.

<u>Definition</u>. A ring topology on a ring R is said to be <u>right linear</u>, if O has a nbd base of opens comprising right ideals of R.
This is equivalent to the set of all open right ideals of R constituting a nbd base for O.

<u>Theorem</u>. Let R be a topological ring and $\mathcal{F}$ the family of all open right ideals of R. If $\mathcal{F}$ is nonempty, then it satisfies the following:

(F1) $\forall A, B \in \mathcal{F}$, $A \cap B \in \mathcal{F}$;
(F2) If $A \in \mathcal{F}$ & $A \subseteq B \leq R_R$, then $B \in \mathcal{F}$;
(F3) If $A \in \mathcal{F}$, then

$$r^{-1}A = \{t \in R : rt \in A\} = \lambda_r^{-1}[A] \in \mathcal{F}.$$

[Omit proof if pressed for time.]

<u>Proof</u> (F1) is an obvious consequence of the fact that the families of open sets, and right ideals, are both closed under finite intersections.

(F2) If $A \in \mathcal{F}$ & $A \subseteq B \leq R_R$, then

$$B = \bigcup_{b \in B} (b + A) = \bigcup_{b \in B} \delta_b[A] \in \mathcal{F}$$

(F3) Let $A \in \mathcal{F}$. Since λ_r is continuous for each $r \in R$, $\lambda_r^{-1}[A]$ is open; it is also a right ideal of R, so $\lambda_r^{-1}[A] \in \mathcal{F}$.

$\square$

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We are thus led to the fundamental definition of this lecture course.

§2 Topologizing filters

Definition. A nonempty family F of right ideals of a ring R that satisfies properties (F1) - (F3) is called a right topologizing filter on R.

Let T be a linear topology on ring R. The family F of all open right ideals of R is a right topologizing filter on R. Since the topology T is linear, F is a nbd base of opens for 0. It follows that T may be recovered from F. Thus distinct linear topologies on R give rise to distinct right topologizing filters on R.

On the other hand, if F is any right topologizing filter on R, then the family of all subsets $\mathcal{O}$ of R of the form $\mathcal{O} = \bigcup_{\delta \in \Delta} (r_\delta + K_\delta)$ where $r_\delta \in R$ and $K_\delta \in F$ $\forall \delta \in \Delta$, can be shown to be a right linear topology on R whose set of open right ideals coincides with F.

We have thus proved:

Theorem. There is a one-to-one correspondence between right linear topologies on a ring R and right topologizing filters on R. □

<u>Notation</u> If R is any ring we shall denote by:

- $\mathrm{Fil}\,R_R$ the set of all right topologizing filters on R, and
- $\mathrm{Id}\,R$ the set of all (two-sided) ideals of R.

<u>Note</u>. If the ring R is commutative, so that all right ideals are two-sided, the Axiom F3 in the definition of a topologizing filter becomes redundant because $r^{-1}A \supseteq A$ for all ideals A, so that F3 is a consequence of F2.

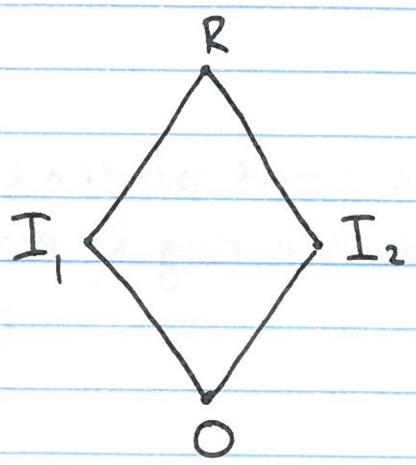
Thus for a commutative ring R, topologizing filters on R are precisely filters, in the lattice sense, on $\mathrm{Id}\,R$.

[Omit one of following examples if pressed for time.]

<u>Examples</u>

(1) Let $R = F_1 \times F_2$ where F_1 & F_2 are any fields. Then

$$\mathrm{Id}\,R = \{0,\ \underset{I_1}{\underset{\parallel}{F_1 \times 0}},\ \underset{I_2}{\underset{\parallel}{0 \times F_2}},\ R\}$$



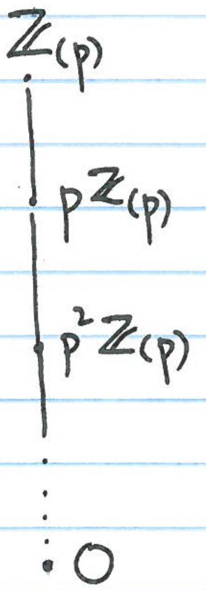
The only topologizing filters on R are:
$\{R\}$, $\{I_1, R\}$, $\{I_2, R\}$, $\mathrm{Id}\,R$.
Note that all these filters are <u>principal</u> (i.e., they possess

smallest members).

(2) Let p be a fixed prime. Consider

$$\mathbb{Z}_{(p)} = \text{localization of } \mathbb{Z} \text{ at prime ideal } p\mathbb{Z}$$
$$= \left\{ \frac{a}{b} \in \mathbb{Q} : a, b \in \mathbb{Z},\ (b,p) = 1 \right\}.$$

$\mathrm{Id}\,\mathbb{Z}_{(p)}$ is a chain:



Each ideal of $\mathrm{Id}\,\mathbb{Z}_{(p)}$ gives rise to a principal filter.

The only non-principal filter is

$$F = \{ p^n \mathbb{Z}_{(p)} : n \in \mathbb{N} \}$$
$$= \text{set of all nonzero ideals of } \mathbb{Z}_{(p)}.$$

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