Lecture 3 — Structure on the set of topologizing filters – $\mathrm{Fil}R_R$ as a lattice

§1 $\mathrm{Fil}R_R$ is a complete lattice

$\mathrm{Fil}R_R$ is partially ordered by inclusion, it has:

- a smallest member $\{R\}$; and
- a largest member comprising the family of all right ideals of R.

It is easily shown that an intersection of right topologizing filters on a ring R is again a right topologizing filter on R. This endows $\mathrm{Fil}R_R$ with the structure of a complete lattice in which meets correspond with intersections.

An explicit description of joins in $\mathrm{Fil}R_R$ requires some preparation.

Let $\mathcal{A}$ be any nonempty family of right ideals in R. The smallest right topologizing filter on R containing $\mathcal{A}$ is given by:

$$\eta(\mathcal{A}) \overset{\text{def}}{=} \{K \leq R_R : K \supseteq \bigcap_{i=1}^{n} r_i^{-1}A_i \text{ for some } r_i \in R \ \& \ A_i \in \mathcal{A},\ 1 \leq i \leq n\}.$$

If $\mathcal{A} = \{A\}$ is a singleton we write $\eta(A)$ in place of $\eta(\{A\})$.

Now let $\{F_\delta : \delta \in \Delta\}$ be a family of right topologizing filters in R. Put $A = \bigcup_{\delta \in \Delta} F_\delta$.

Using Axioms F2 and F3 the intersection $\bigcap_{i=1}^{n} r_i^{-1} A_i$ appearing in the above formula for $\eta(A)$, can be simplified to $\bigcap_{\delta \in \Delta'} A_\delta$ where Δ' is some finite subset of Δ and $A_\delta \in F_\delta \ \forall \delta \in \Delta'$.

This yields the following formula for joins in $\mathrm{Fil}\, R_R$:

$$\bigvee_{\delta \in \Delta} F_\delta = \eta\Big(\bigcup_{\delta \in \Delta} F_\delta\Big)$$

$$= \{K \leq R_R : K \supseteq \bigcap_{\delta \in \Delta'} A_\delta \text{ for some finite } \Delta' \subseteq \Delta \ \& \ A_\delta \in F_\delta \ \forall \delta \in \Delta'\}.$$

In the subsections that follow, we shall establish the following lattice theoretic properties for $\mathrm{Fil}\, R_R$:

<u>Summary of properties of the complete lattice $\langle \mathrm{Fil}\, R_R ; \cap, \vee \rangle$</u>:

- Modular [vdB (PhD, 1995); Raggi, Ríos Montes, Wisbauer (2000)].
- Atomic [Golan, 1987].
- Compact (& therefore coatomic) [Golan, 1987].
- compactly generated (& therefore upper continuous) [Golan, 1987].

- Uniquely pseudo-complemented [Raggi, Ríos Montes, Wisbauer (2000)].

§2 Atoms in $\mathrm{Fil}\,R_R$

A nonzero element e of a complete lattice L is called an atom if $\forall a \in L$, $a \leq e \Rightarrow a = 0$ or $a = e$. The lattice L is called atomic if $\forall$ nonzero $a \in L$ $\exists$ atom $e \in L$ s.t. $0 < e \leq a$.

Theorem The following statements are equivalent for $F \in \mathrm{Fil}\,R_R$:

(1) F is an atom of $\mathrm{Fil}\,R_R$;

(2) $F = \eta(H)$ for some maximal right ideal H of R.

[Prove only (1) ⇒ (2) if pressed for time.]

Proof (1) ⇒ (2). Since $F \neq \{R\}$ it must contain a proper right ideal which, in turn, must be contained in some maximal right ideal H, say. (This can be established using Zorn's Lemma.) By Axiom F2, $H \in F$, so $\eta(H) \subseteq F$. Since F is an atom this entails $F = \eta(H)$.

(2) ⇒ (1). Let $G \in \mathrm{Fil}\,R_R$ with $\eta(H) \supseteq G \neq \{R\}$. Since G is nontrivial it must contain a proper right ideal of R, and thus a maximal right ideal of R, say L. Since $L \in \eta(H)$, $L \supseteq \bigcap_{i=1}^{n} r_i^{-1} H$

for some $r_i \in R$, $1 \leq i \leq n$.
Note that

$$R/\bigcap_{i=1}^{n} r_i^{-1}H \longleftrightarrow \prod_{i=1}^{n} R/H$$

$$r + \bigcap_{i=1}^{n} r_i^{-1}H \longmapsto (rr_1 + H, \ldots, rr_n + H).$$

Observe also that R/L is an epimorphic image of $R/\bigcap_{i=1}^{n} r_i^{-1}H$. Since $\prod_{i=1}^{n} R/H$ is a semisimple right R-module (it is a direct sum of n-copies of the simple module R/H), R/L is isomorphic to a submodule of $\prod_{i=1}^{n} R/H$. Since R/L is also simple ($\because L$ is a maximal right ideal) this entails $R/L \cong R/H$. It follows that $H = t^{-1}L$ for some $t \in R$ whence $H \in \eta(L)$ and $\eta(H) \subseteq \eta(L) \subseteq G$. Equality follows. □

The proof of (1) $\Rightarrow$ (2) in the above theorem shows that every non-trivial $F \in \mathrm{Fil}\, R_R$ contains a maximal proper right ideal H, and from this it follows that $F \supseteq \eta(H)$. Thus F contains an atom of $\mathrm{Fil}\, R_R$. We have thus proved the following

<u>Theorem</u>. $\mathrm{Fil}\, R_R$ is atomic. □

§3 Compact elements in $\mathrm{Fil}\,R_R$

An element c of a complete lattice L is said to be compact if $\forall X \subseteq L$, if $c \leq \bigvee X$, then $c \leq \bigvee X'$ for some finite subset X' of X.
We call L a compact lattice if its top element 1, is compact.
We call L compactly generated (or algebraic) if every element in L can be expressed as a join of compact elements.

The subobject lattices of most algebraic structures, e.g. groups & rings, are compactly generated.

Theorem. The following statements are equivalent for $\mathcal{F} \in \mathrm{Fil}\,R_R$:
(1) $\mathcal{F}$ is compact;
(2) $\mathcal{F} = \eta(L)$ for some $L \leq R_R$.

[Omit proof if pressed for time.]

Proof (1) ⇒ (2). Suppose $\mathcal{F}$ is compact. Write
$\mathcal{F} = \bigvee_{K \in \mathcal{F}} \eta(K)$. Inasmuch as $\mathcal{F}$ is compact,
$\mathcal{F} = \bigvee_{K \in \mathcal{F}'} \eta(K)$ for some finite subfamily $\mathcal{F}'$ of $\mathcal{F}$.
Put $L = \bigcap \mathcal{F}'$. Since L is a finite intersection of right ideals in $\mathcal{F}$, $L \in \mathcal{F}$, whence $\mathcal{F} \supseteq \eta(L)$.
On the other hand, since $L \subseteq K$, $\forall K \in \mathcal{F}'$, we

must have $\eta(L) \supseteq \eta(K)\ \forall K \in F'$, whence $\eta(L) \supseteq \bigvee_{K \in F'} \eta(K) = K$. We conclude that $F = \eta(L)$.

<u>(2) ⇒ (1)</u>. Suppose $F = \eta(L) \subseteq \bigvee_{\delta \in \Delta} F_\delta$ where $\{F_\delta : \delta \in \Delta\} \subseteq \mathrm{Fil}\,R_R$. Then $L \in \bigvee_{\delta \in \Delta} F_\delta$, so $L \supseteq \bigcap_{\delta \in \Delta'} K_\delta$ for some finite $\Delta' \subseteq \Delta$ with $K_\delta \in F_\delta$ $\forall \delta \in \Delta'$. It follows that $L \in \bigvee_{\delta \in \Delta'} F_\delta$, whence $F = \eta(L) \subseteq \bigvee_{\delta \in \Delta'} F_\delta$. We conclude that F is compact.

□

<u>Notes</u> (1) The top element of $\mathrm{Fil}\,R_R$ is $\eta(0)$ which is compact, so $\mathrm{Fil}\,R_R$ is a compact lattice.

(2) An application of Zorn's Lemma shows that compact lattices are coatomic (dual notion to atomic).

<u>Theorem</u>. $\mathrm{Fil}\,R_R$ is compactly generated.

<u>Proof</u>. Take $F \in \mathrm{Fil}\,R_R$. Then $F = \bigvee_{K \in F} \eta(K)$ which expresses F as a join of compact elements □

§4 <u>Pseudo-complements in $\mathrm{Fil}\,R_R$</u>

Let L be a complete lattice and $a \in L$. An element a' of L is called a <u>pseudo-complement</u> of a in L if $a' \wedge a = 0$ & a' is maximal with this property. We call the lattice L <u>uniquely pseudo-complemented</u> if every $a \in L$ has a unique pseudo-complement.

<u>Theorem</u>. Fil R_R is uniquely pseudo-complemented. □

[A description of pseudo-complements in Fil R_R requires module theoretic methods that lie beyond the scope of this lecture course.]

<u>Theme</u>. Determine the extent to which information about a ring R and its category of modules is encoded in the structure of Fil R_R.

Here are some examples of results in the spirit of this theme.

<u>Theorem</u> [Folklore]. The following assertions are equivalent for a ring R:

(1) Fil R_R is trivial, i.e., Fil R_R contains only $\{R\}$ and $\eta(0)$.

(2) R is simple artinian, so $R \cong M_n(D)$ for some $n \in \mathbb{N}$ & division ring D.

[Omit if pressed for time.]

<u>Proof</u> <u>(1) ⇒ (2)</u>. (only) Since Fil R_R is atomic, $\eta(0) = \eta(H)$ for some maximal right ideal H of R.

Thus $0 \in \eta(H)$, so $0 = \bigcap_{i=1}^{n} r_i^{-1}H$ for some $r_i \in R$, $1 \le i \le n$.

It follows that

$$R_R \cong R/0 = R/\bigcap_{i=1}^{n} r_i^{-1}H \longleftrightarrow \prod_{i=1}^{n} R/H$$

$$r + \bigcap_{i=1}^{n} r_i^{-1}H \longmapsto (rr_1+H, \dots, rr_n+H).$$

Since $\prod_{i=1}^{n} R/H$ is a semisimple right R-module, $R_R \cong \prod_{i=1}^{m} R/H$ for some $m \le n$.

Then $R \cong \operatorname{End} R_R \cong \operatorname{End}\left(\prod_{i=1}^{m} R/H\right)$

$\cong \mathbb{M}_m(\operatorname{End} R/H)$

$= \mathbb{M}_m(D)$ where $D = \operatorname{End} R/H$ is a Division ring by Schur's Lemma. □

The above result can be extended to:

<u>Theorem</u> [Raggi et al] The following statements are equivalent for a ring R:

(1) $\operatorname{Fil} R_R$ is a Boolean lattice;

(2) R is semisimple, so $R \cong \mathbb{M}_{n_1}(D_1) \times \dots \times \mathbb{M}_{n_k}(D_k)$ for $n_1, \dots, n_k \in \mathbb{N}$ and division rings $D_1, \dots, D_k$.

□

Question [J.E. Viola-Prioli, PhD, 1975]

When is $\mathrm{Fil}\,R_R$ linearly ordered, i.e., a chain?

- A sufficient condition: R is a right chain ring (a ring whose lattice of right ideals is a chain). This condition is not necessary, however!

Question

When is $\mathrm{Fil}\,R_R$ a distributive lattice?

- If the ring R is commutative, then $\mathrm{Fil}\,R_R$ is distributive iff $\mathrm{Id}\,R$ is distributive. If R is a commutative noetherian domain, then $\mathrm{Id}\,R$ is distributive iff R is a Dedekind domain; the class of such rings includes PIDs.

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